A
TREATISE ON PROBLEMS
OF
MAXIMA AND MINIMA,
SOLVED BY ALGEBRA.

BY

RAMCHUNDRA,
LATE TEACHER OF SCIENCE, DELHI COLLEGE.

REPRINTED BY ORDER OF THE HONOURABLE COURT OF DIRECTORS OF THE EAST-INDIA COMPANY FOR CIRCULATION IN EUROPE AND IN INDIA, IN ACKNOWLEDGMENT OF THE MERIT OF THE AUTHOR, AND IN TESTIMONY OF THE SENSE ENTERTAINED OF THE IMPORTANCE OF INDEPENDENT SPECULATION AS AN INSTRUMENT OF NATIONAL PROGRESS IN INDIA.

Under the Superintendence of

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EDITOR'S PREFACE.

In the year 1850, my friend the late* J. E. Drinkwater-Bethune forwarded to England a number of copies of a work on

* John Elliot Drinkwater, the eldest son of Lieutenant-Colonel John Drinkwater, author of the "History of the Siege of Gibraltar," was born July 12, 1801, was educated at Westminster and at Trinity College, Cambridge, and took the degree of B.A., as fourth wrangler, in 1823. He was called to the bar about 1827, and when Lord Grey came into office in 1831, was employed by the Government on various commissions. He was for about fourteen years counsel to the Home Office, and had much to do with the Parliamentary Reform Bill, the Municipal Reform Bill, Medical Reform Bills, the establishment of the Queen's Colleges in Ireland, the organization of the County Courts, and other important measures. During this time he published, in the Library of Useful Knowledge, part of a treatise on algebraical expressions, which was never finished, and lives of Galileo and Kepler, which exhibit great research and acumen, and stand high among modern English efforts in scientific biography. He also printed for private circulation a translation of Schiller's "Maid of Orleans," and of some of Tegner's Swedish poems. In 1836 his mother inherited the estate of Bethune of Balfour, in Fife, and the whole family added the name of Bethune to their surname. In 1848 he sailed for India as fourth ordinary member of the Supreme Council, and president of the Law Commission. Lord Dalhousie added the presidentship of the Council of Education. He devoted more attention to the cause of education than even to his legislative duties. In his private capacity he founded at Calcutta a school for Hindoo girls of the higher classes. He bequeathed the land and building to the East-India Company, on condition that the school should become a Government institution. The offer was accepted, and the school is in successful operation. He also prevailed upon the representatives of the family of Tippoo Saib to throw open to Mohamedans of good family the school which had been endowed for the exclusive use of that family. He procured an enactment by which natives converted to Christianity are not deprived of their rights of inheritance. He had to encounter much virulence of opposition, both from natives and Europeans; but his character and manners

Maxima and Minima, by Ramchundra, teacher of science, Delhi College, with directions to present copies to various persons, and among others to myself. On examining this work I saw in it, not merely merit worthy of encouragement, but merit of a peculiar kind, the encouragement of which, as it appeared to me, was likely to *promote native effort towards the restoration of the native mind in India.* Mr. Drinkwater-Bethune's lamented death, which took place shortly after he had dispatched the books, prevented my knowing whether he also entertained any opinion similar to mine as to the distinctive character of Ramchundra's work; but, from his own knowledge of the history of mathematics, I think it highly probable. I addressed my thanks for the present to his successor, Mr. Colvile, with some remarks on the subject. Having taken further time to think of it, I determined to call the attention of the Court of Directors to Ramchundra's work, in the hope that it would lead to acknowledgment of his deserts. I accordingly addressed a letter (July 24, 1856) to Colonel Sykes, the Chairman, to whom I had previously mentioned the matter at a casual meeting. This letter was at once forwarded to the Lieutenant-Governor of the North-West Provinces, with instructions to procure a report on the case, and to suggest, on the supposition of a public reward being approved of, the kind of reward which should be given. Answers were received by the Court, which were communicated to me on the 3rd of March, 1858. They contained various replies to the questions proposed, by H. Stewart Reid, Esq., Director of Public Instruction in the North-West Provinces, and other gentlemen connected with the

procured him the respect he deserved before his death, which took place August 12th, 1851, from inflammation of the liver. He lived two lives of real utility, one in England and one in India; and as many in either country know nothing of his career in the other, and this work is intended for both, a short abstract of his life is here given. This abstract is the more appropriate as his encouragement of Ramchundra was the first of the train of circumstances which produced the reprint now before the reader.

same department. With some difference of opinion as to the mode of acknowledgment, there was unanimous appreciation of Ramchundra's services to his country, and admission of the desirableness of encouraging his efforts. The Court accompanied the communication of these answers to me with a request that I would point out how to bring Ramchundra under the notice of scientific men in Europe. In my reply (March 18), assuming distinctly that I conceived the question to be, not merely how Ramchundra could be rewarded, but how his work might be made most effective in the development of Hindoo talent, I recommended the circulation of the work in Europe, with a distinct account of the grounds on which the step was taken. I entered at some length into my own view of those grounds, and volunteered to draw up the statement which should accompany the publication. After some correspondence on details, the Court (July 1), expressing entire satisfaction with my views, and characterizing them as "deserving of the most attentive consideration by all who are charged with the superintendence of education in India in its higher grades," accepted my offer to superintend the present reprint, for circulation in Europe and in India.

I shall at once proceed to a short account of these views; after which I shall give some account of Ramchundra, the author of the work. Of course it will be remembered that the late Court of Directors is in no way answerable for the details of my exposition, though their decided approbation was bestowed on the general sketch which I laid before them.

There are many persons, even among those who seriously turn their thoughts to the improvement of India, who look upon the native races as men to be dealt with in the same manner as Caffres or New Zealanders. Judging by the lower races of the Peninsula, and judging even these more by the grosser parts of their mythology than by the state of domestic life and hereditary institutions, they presume that the Indian question resolves itself into an

inquiry how to create a mind in the country, and that mind fashioned on the English standard. They forget that at this very moment there still exists among the higher castes of the country —castes which exercise vast influence over the rest—a body of literature and science which might well be the nucleus of a new civilization, though every trace of Christian and Mohamedan civilization were blotted out of existence. They forget that there exists in India, under circumstances which prove a very high antiquity, a philosophical language which is one of the wonders of the world, and which is a near collateral of the Greek, if not its parent form. From those who wrote in this language we derive our system of arithmetic, and the algebra which is the most powerful instrument of modern analysis. In this language we find a system of logic and of metaphysics: an astronomy worthy of comparison with that of Greece in its best days; above comparison, if some books of Ptolemy's Syntaxis be removed. We find also a geometry, of a kind which proves that the Hindoo was below the Greek as a geometer, but not in that degree in which he was above the Greek as an arithmetician. Of the literature, poetry, drama, &c., which flourished in union with this science, I have not here to speak.

Those who consult Colebrooke's translation of the *Vija Ganita*, or the account given of it in the *Penny Cyclopædia*, will see that I have not exaggerated the point most connected with this preface. For others I will quote the impression made, five-and-thirty years ago, upon the mind of a mathematician whose subsequent career and present position will give that weight to an extract from his opinions which would have been given to any reader of the whole article by the article itself, even had it been anonymous. Sir John Herschel, in the historical article *Mathematics*, in *Brewster's Cyclopædia*, after some general account of the Hindoo algebra, proceeds as follows:—

"The Brahma Sidd'hanta, the work of Brahmegupta, an Indian astronomer at the beginning of the seventh century, contains a general method for the resolution of indeterminate problems of the second degree; an investigation which actually baffled the skill of every modern analyst till the time of Lagrange's solution, not excepting the all-inventive Euler himself. This is matter of a deeper dye. The Greeks cannot for a moment be thought of as the *authors* of this capital discovery; and centuries of patient thought, and many successive efforts of invention, must have prepared the way to it in the country where it did originate. It marks the maturity and vigour of mathematical knowledge, while the very work of Brahmegupta, in which it is delivered, contains internal evidence that in his time geometry at least was on the decline. For example, he mentions several properties of quadrilaterals as general, which are only true of quadrilaterals inscribed in a circle. The discoverer of these properties (which are of considerable difficulty) could not have been ignorant of this limitation, which enters as an essential element in their demonstration. Brahmegupta then, in this instance, retailed, without fully comprehending, the knowledge of his predecessors. When the stationary character of Hindu imtelligence is taken into the account, we shall see reason to conclude, that all we now possess of Indian science is but part of a system, perhaps of much greater extent, which existed at a very remote period, even antecedent to the earliest dawn of science among the Greeks, and might authorize as well the visits of sages as the curiosity of conquerors."

Greece and India stand out, in ancient times, as the countries of indigenous speculation. But the intellectual fate of the two nations was very different. Among the Greeks, the power of speculation remained active during their whole existence as a nation, even down to the taking of Constantinople: it declined, indeed, but it was never extinguished. Their latest knowledge was inquisitive,

as well as their earliest. They preserved their great writers unabridged and unaltered; and Euclid did not degenerate into what are called *practical rules.*

In India, speculation died a natural death. A taste for *routine* —a thing to which inaccurate thinkers give the name of *practical* —converted their system into a collection of rules and results. Of this character are all the mathematical books which have been translated into English; perhaps all which still exist. That they must have had an extensive body of demonstrated truths is obvious; that they lost the power and the wish to demonstrate is certain. The Hindoo became, to speak of the highest and best class, the teacher of results which he could not explain, the retailer of propositions on which he could not found thought. He had the remains of ancestors who had investigated for him, and he lived on such comprehension of his ancestors as his own small grasp of mind would allow him to obtain. He fed himself and his pupils upon the chaff of obsolete civilization, out of which Europeans had thrashed the grain for their own use.

But the mind thus degenerated is still a mind; and the means of restoring it to activity differ greatly from those by which a barbarous race is to be gifted with its first steps of progress. No man alive can, on sufficient data, reason out the restoration of a decayed national intellect, possessed of a system of letters and science which has left nothing but dry results, inveterate habits of routine, great reverence for old teachers, and small power of comprehending the very teaching which is held in traditional respect. And this because the question is now tried for the first time. Many friends of education have proposed that Hindoos should be fully instructed in English ideas and methods, and made the media through which the mass of their countrymen might receive the results in their own languages. Some trial has been given to this plan, but the results have not been very encouraging, in any of the higher branches of knowledge. My conviction is,

that the Hindoo mind must work out its own problem; and that all we can do is to *set it to work*; that is, to promote independent speculation on all subjects by previous encouragement and subsequent reward. This is the true plan; all others are neither fish nor flesh.

That sound judgment which gives men well to know what is best for them, as well as that faculty of invention which leads to development of resources and to the increase of wealth and comfort, are both materially advanced, perhaps cannot rapidly be advanced without, a great taste for pure speculation among the general mass of the people, down to the lowest of those who can read and write. England is a marked example. Many persons will be surprised at this assertion. They imagine that our country is the great instance of the refusal of all *unpractical* knowledge in favour of what is *useful*. I affirm, on the contrary, that there is no country in Europe in which there has been so wide a diffusion of speculation, theory, or what other unpractical word the reader pleases. In our country, the scientific *society* is always formed and maintained by the people; in every other, the scientific *academy*—most aptly named—has been the creation of the government, of which it has never ceased to be the nursling. In all the parts of England in which manufacturing pursuits have given the artisan some command of time, the cultivation of mathematics and other speculative studies has been, as is well known, a very frequent occupation. In no other country has the weaver at his loom bent over the *Principia* of Newton; in no other country has the man of weekly wages maintained his own scientific periodical. With us, since the beginning of the last century, scores upon scores—perhaps hundreds, for I am far from knowing all—of annuals have run, some their ten years, some their half-century, some their century and a half, containing questions to be answered, from which many of our examiners in the universities have culled materials for the academical contests. And these questions have

always been answered, and in cases without number by the lower order of purchasers, the mechanics, the weavers, and the printers' workmen. I cannot here digress to point out the manner in which the concentration of manufactures, and the general diffusion of education, have affected the state of things; I speak of the time during which the present system took its rise, and of the circumstances under which many of its most effective promoters were trained. In all this there is nothing which stands out, like the state-nourished academy, with its few great names and brilliant single achievements. This country has differed from all others in the wide diffusion of the disposition to speculate, which disposition has found its place among the ordinary habits of life, moderate in its action, healthy in its amount.

The history of England, as well as of other countries, having impressed me with a strong conviction that pure speculation is a powerful instrument in the progress of a nation, and my own birth and descent having always given me a lively interest in all that relates to India, I took up the work of Ramchundra with a mingled feeling of satisfaction and curiosity: a few minutes of perusal added much to both. I found in this dawn of the revival of Hindoo speculation two points of character belonging peculiarly to the Greek mind, as distinguished from the Hindoo; one of which may have been fostered by the author's European teachers, but certainly not the other.

The first point is a leaning towards geometry. Persons who are not mathematicians imagine that all mathematicians are for all mathematics. Nothing can be more erroneous. Not merely have the two great branches, geometry and algebra, their schools of disciples, each of which looks coldly upon the other; but even geometry itself, and algebra itself, have subdivisions of which the same thing may be said. For example, Mr. Drinkwater-Bethune, above mentioned, was by taste an *algebraist;* as a practised eye would at once detect from his unfinished work on equations.

Business brought him to my house one morning, nearly thirty years ago, at a time when I happened to be studying some of the geometrical developments of the school of Monge. On my pointing out to him some of the most remarkable of the conclusions, he said, with a smile, "I see that sort of thing has charms for you." Now the Hindoo was also an algebraist, as decidedly as the Greek was a geometer: the first sought refuge from geometry in algebra, the second sought refuge from arithmetic in geometry. The greatness of Hindoo invention is in algebra; the greatness of Greek invention is in geometry. But Ramchundra has a much stronger leaning towards geometry than could have been expected by a person acquainted with the *Vija Ganita;* but he has not the power in geometry which he has in algebra. I have left one or two failures—one very remarkable—unnoticed, for the reader to find out. Should this preface—as I hope it will—fall into the hands of some young Hindoos who are systematic students of mathematics, I beg of them to consider well my assertion that their weak point must be strengthened by the cultivation of pure geometry. Euclid must be to them what Bhascara, or some other algebraist, has been to Europe.

The second point is yet more remarkable. Greek geometry, as all who have read Euclid may guess, gained its strength by *striving against self-imposed difficulties.* It was not permitted to take instruments from every conception which the human mind could form; definite limitation of means was imposed as a condition of thought, and it was sternly required that every feat of progress should be achieved by those means, and no more. Just as the Greek architecture studied the production of rich and varied effect out of the simplest elements of form, so the Greek geometry aimed at the demonstration of all the relations of figure on the smallest amount of postulated basis. The great problem of squaring the circle, now with good reason held in low esteem, was the struggle of centuries to bring under the

dominion of the prescribed means what might with the utmost ease have been conquered by a very small additional allowance. The attempt was unsuccessful; so was that of Columbus to discover India from the west. But Columbus commenced the addition of America to the known world; and in like manner the squarers of the circle, and their refuters, added field on field to the extent of geometry, and aided largely in the preparation for the modern form of mathematics. Very few of these additions would have been made, at or near the time when they were made, if it had satisfied the Greek mind to meet each difficulty, as it occurred, by permission to use additional assumptions in geometry.

The remains of the Hindoo algebra and geometry show to us no vestige of any attempt to gain force of thought by struggling against limitation of means: this, of course, because their mode of demonstration does not appear in the works which are left, or at least in those which have become known to Englishmen. But we have here a native of India who turns aside, at no suggestion but that of his own mind, and applies himself to a problem which has hitherto been assigned to the differential calculus, under the condition that none but purely algebraical process shall be used. He did not learn this course of proceeding from his European guides, whose aim it has long been to push their readers into the differential calculus with injurious speed, that they may reach the full application of mathematics to physics; and who often allow their pupils to read Euclid with eyes shut to his limitations. Ramchundra proposed to himself a problem which a beginner in the differential calculus masters with a few strokes of the pen in a month's study, but which might have been thought hardly within the possibilities of pure algebra. His victory over the theory of the difficulty is complete. Many mathematicians of sufficient power to have done as much would have told him, when he first began, that the end proposed was perhaps unattainable by any amount of thought; next, that when attained, it would be of no use. But he found in

the demands of his own spirit an impulse towards speculation of a character more fitted to the state of his own community than the imported science of his teachers. He applied to the branch of mathematics which is indigenous in India, the mode of thought under which science made its greatest advances in Greece. My own strong suspicion that it was the want of this mode of thought which allowed the decline of algebra in ancient India, coupled with my thorough conviction that, whether or no, this mode of thought yields the proper nutriment for mathematical science in its early and feeble life, produced the recommendation to the Court of Directors to which this reprint owes its existence.

Ramchundra's problem—and I think it ought to go by that name, for I cannot find that it was ever current* as an exercise of ingenuity in Europe—is to find the value of a variable which will make an algebraical function a maximum or a minimum, under the following conditions. Not only is the differential calculus to be excluded but even that germ of it which, as given by Fermat in his treatment of this very problem, made some think that he was entitled to clain the invention. The values of ϕx and of $\phi(x+h)$ are not to be compared; and no process is to be allowed which immediately points out the relation of ϕx to the derived function $\phi' x$. A mathematician to whom I stated the conditioned problem made it, very naturally, his first remark, that he could not see how on earth I was to find out when it would be biggest, if I would not let it grow. The mathematician will at last see that the question resolves itself into the following:—Required a constant, r, such that $\phi x - r$ shall have a pair of equal roots, without assuming the development of $\phi(x+h)$, or any of its consequences.

* It would not at all surprise me if it should be found that some one inquirer has suggested the problem; but, if so, I think the search which I have made entitles me to say that the suggestion entirely failed to attract attention, and to establish the difficulty as a recognized exercise.

It will readily be seen that a short paper, with a few examples, would have sufficed to put the whole matter before a scientific society. But it was Ramchundra's object to found an elementary work upon his theorem, for the use of beginners, with a large store of examples. As to the method which he has adopted, Europeans must remember that his purpose is to teach Hindoos, and that probably he knows better how to do this than they could tell him. The excessive reiteration of details, and the extreme minuteness of the algebraical manipulations, are excellent examples of that patience of routine which is held to be a part of the Hindoo character. I may make two remarks on matters which would strike the most casual observer.

First, the constant occurrence of "the same solved without impossible roots," and the transformation by which it is effected, will remind the English mathematician who has his half-century over his head, of the old "pure quadratic," and the victory which was supposed to be gained when the "adfected quadratic" was evaded by attention to the structure of the given equation. Ramchundra and Dr. Miles Bland, &c. &c., are here precisely on the same scent, both making much of the same little.

Secondly, in the confusion of terms which sometimes appears, in language implying that an *equation* is a factor of an *equation*, instead of an *expression* a factor of an *expression*, we have the same incorrectness which appears in more than one edition of Waring's *Meditationes Algebraicæ*, and which occasioned some amount of objection to the whole theory from those who could not see the inaccuracy and its correction. As in

"Scribatur $x-\alpha=0$, $x-\beta=0$, $x-\gamma=0$, $x-\delta=0$, &c. et per æquationem ex horum factorum continuâ multiplicatione

$$(x-\alpha)\times(x-\beta)\times(x-\gamma)\times(x-\delta)\times \&c.=0$$

generatam dividatur data æquatio."

I believe that selections from Ramchundra's work might advan-

tageously be introduced into elementary instruction in this country. The exercise in quadratic equations which it would afford, applied as it is to real problems, would advantageously supersede some of the conundrums which are manufactured under the name of problems producing equations.

In the printing I have followed the original in every point, altering nothing except obvious errata, including the restoration of the numeral symbol 0, which in the original is always the letter *o*. This again is a mistake into which Waring allowed his printer to fall in almost all his writings. I thought that the European reader would be more curious to look at the way in which the Calcutta printer treated mathematical manuscript when his author was no nearer than Delhi, than to see the manner in which I could mend it. My printers, Messrs. Cox & Wyman, have entered fully into the plan, and have produced as nearly a facsimile as possible. I may add that the Calcutta printer has acquitted himself in a manner entitled to especial notice and high praise.

Ramchundra, the author of this work, has transmitted to me some notes of his own life, from which I collect as follows. He was born in 1821, at Paneeput, about fifty miles from Delhi. His father, Soondur Lall, was a Hindoo Kaeth, and a native of Delhi, and was there employed under the collector of the revenue. He died at Delhi in 1831-32, leaving a widow (who still survives) and six sons. After some education in private schools, Ramchundra entered the English Government school at Delhi, to every pupil of which two rupees a month were given, and a scholarship of five rupees a month to all in the first and second classes. In this school he remained six years. It does not appear that any particular attention was paid to mathematics in this school; but, shortly before leaving it, a taste for that science developed itself in Ramchundra, who studied at home with such books as he could procure. After leaving school, he obtained employment as

a writer for two or three years. In 1841, changes took place in the educational department of the Bengal presidency; the school was formed into a college; and Ramchundra obtained, by competition, a senior scholarship, with thirty rupees a month. In 1844, he was appointed teacher of European science in the Oriental department of the college, through the medium of the vernacular, with fifty rupees a month additional. A vernacular translation society was instituted, and Ramchundra, in aid of its object, translated or compiled works in Oordoo, and also on algebra, trigonometry, &c., up to the differential and integral calculus. "These translations"—I now proceed to quote Ramchundra's words—"were introduced into the Oriental department as class-books; so that in two or three years many students in the Arabic and Persian departments were, to a certain extent, acquainted with English science: and the doctrines of the ancient philosophy, taught through the medium of Arabic, were cast into the shade before the more reasonable and experimental theories of modern science. The old dogmas, such as 'that nature abhors a vacuum,' and 'that the earth is the fixed centre of the universe,' were generally laughed at by the higher students of the Oriental, as well as by those of the English departments of the Delhi College. But the learned moulwees, &c., who lived in the city and had no connection with the college, did not like this innovation on their much-beloved theories of the ancient Greek philosophy, which from centuries past had been cultivated among them.

"I, with the assistance of the higher students of the English and Oriental departments, formed a society for the diffusion of knowledge among our countrymen. We were ambitious enough to imitate the plan of the *Spectator*. We first commenced a monthly, and then a bi-monthly periodical, called the *Fawáedánnázireen* (*i. e.* useful to the reader), at the cheap price of four annas a month, in which notices of English science were given, and in which not only were the dogmas of the Mohamedan and Hindoo

philosophy exposed, but also many of the Hindoo superstitions and idolatries were openly attacked. The result of this was that many of our countrymen, the Hindoos, condemned us as infidels and irreligious; but as we did not advocate Christianity, but only recommended a kind of deism, and as we never lost our caste publicly, by eating and drinking, all our free discussions did not much alarm our Hindoo friends. When in private meetings our friends, seeing us so warmly advocating English science and knowledge, taunted us by saying we will become Christians, as such and such pundit had become, then we considered this as an insult, and stated in reply, that the pundit referred to had not received any English education, and that he was ignorant, and was therefore deceived by the missionaries, whom we considered as ignorant and superstitious as our own uneducated friends. We went so far as to challenge our Hindoo friends to bring any Christian missionary to us, and see whether he can persuade us. It was then my conscientious belief that educated Englishmen were too much enlightened to believe in any bookish religion except that of reason and conscience, or deism. Sometimes, when the late Baptist missionary, Mr. Thompson, stopped me in the bazaar, and required me to think of my eternal concerns, and gave me some tracts, &c. in Persian and Oordoo, I did not speak to him much,—received parts of the New Testament, &c., and when I returned home I put them in a corner, and never read them.

"Once a learned Mohamedan came to me with a copy of the New Testament in Oordoo, and having read some portion of St. Paul's epistles, spoke greatly against the apostle, and the missionaries in general, because St. Paul teaches that circumcision is of no use for salvation. His object in reading this to me was to get an English scholar and a teacher of English science to agree with him in saying how absurd Christianity and Christians were. Though what he read was in my mother tongue, still it was wholly Greek to me; I did not understand the question. In

order to put a stop to this talk, in which I had then no interest, I briefly told him that, for my part, I considered not only Christianity, but also Mohamedanism, and all bookish religions, as absurd and false. Upon this all Hindoos and Mohamedans present paid me the compliment of being a philosopher, and departed with marks of approbation and goodwill.

"A respectable and learned Mohamedan, secretly assisted by some other celebrated moolwees of the city, published a treatise in Oordoo in refutation of the motion of the earth, on the principles of Aristotelian philosophy; the whole train of reasonings being copied almost verbatim from a metaphysical work in Arabic, called *Myboodee*. But no sooner was this publication made over to us, than a moolwee, and some higher students of the Arabic department, got up a sharp reply, and published it; to which no answer was returned. Afterwards, in addition to the bi-monthly periodical, we commenced a monthly magazine, called the *Moohib-i-Hind*, or the *Friend of India*. But it must be confessed that we did not receive sufficient support from the native public, and it was principally through the patronage of English authorities, as Sir John Lawrence (the magistrate of Delhi), Mr. A. A. Roberts (ditto ditto), Dr. A. Ross, Mr. J. F. Gubbins (then judge at Delhi), who subscribed for several copies of our periodicals, that we got sufficient money to pay the expenses of our publications. But afterwards, times and circumstances being changed, we were compelled to discontinue them; so that, in 1852, the bi-monthly periodical was also discontinued, after being kept up more than five years.

"In 1850 I published the mathematical work to which this account of my humble life is intended to be attached. As the work was published in Calcutta, I requested a friend of mine there to present copies of it to distinguished men in that city; but the reviews published in some Calcutta papers were generally unfavourable to the publication." In another letter Ramchundra

says, "When I composed my work on 'Problems of Maxima and Minima,' I built many castles in the air; but Calcutta reviewers, &c. destroyed these empty phantasms of my brain." He also describes himself as subjected to kind rebukes from some of the best friends of native education in the North-West Provinces, for his ambition in publishing his work in *English*.

"During the examination vacation in 1851, having obtained three months' leave from the college, I went down to Calcutta, of which I had heard much, and which I was very desirous of seeing. When I arrived there, I happened to read a number of the *Calcutta Review*, in which very unfavourable notice was given of my work. My friends then advised me to write an answer to it, which I did, and the editor of the *Englishman* very kindly published it in his paper.

"Dr. Sprenger, who was formerly principal of the Delhi College, introduced me to the Honourable D. Bethune, of the Supreme Council, who very kindly received from me thirty-six copies of my work, and paid me 200 rupees as a donation." It should be noted that Ramchundra had published the work entirely at his own expense. "I afterwards learned that he sent a number of these copies to England."

After mention of the correspondence, &c. described at the beginning of this Preface, Ramchundra proceeds as follows:—

"The honourable members of the Court of Directors were pleased to confer honours upon me, and the Government in this country sanctioned a khillut (dress of honour) of five pieces, which I am told I will obtain at Delhi, and also a reward of 2,000 rupees, which I have already received at the hands of Captain Robert Maclagan. I am much thankful to the English Government that they are so bent upon encouraging science and knowledge among the natives of this country, as to take notice of a poor native of Delhi like myself.

"The most important event of my life, at least what I consider

to be as such, was, that by God's unsearchable and gracious Providence I was brought to the knowledge of the Saviour. After I had finished my mathematical work, and before I went down to Calcutta on leave, I had become a believer in the Gospel. Before this belief had taken possession of my heart, there were two erroneous notions in my head (and which I believe must ever be in the heads of nearly all native youths educated in Government colleges and schools, as long as the system of instruction continues to be pursued as it is till now)." The first of these notions was that the English themselves did not believe in Christianity, because they did not, as a Government, exert themselves to teach it. The second was that a person who believes in one God stands in need of no other religion. I omit the details of Ramchundra's reasoning, because this publication is expressly intended for India as well as England, and because I do not feel authorized to introduce into a work published by the late and present Government of India, what might originate a discussion on a most difficult question of Indian policy. Ramchundra proceeds thus:—

"Both of these erroneous notions were dispelled in the following manner. Once a Brahmin student was sent by an English officer from Kotah to the Delhi College, and was recommended to the principal's notice. This stranger in Delhi waited to see the church during divine service. The principal, Mr. Taylor, also requested me to go with the Brahmin student to see the divine service in the church, if I liked. And thus, out of mere curiosity, we went there, and saw several English gentlemen whom I respected as well-informed and enlightened persons. Many of them kneeled down, and appeared to pray most devoutly. I was thus undeceived of my first erroneous notion, and felt a desire to read the Bible. Mr. Taylor recommended me first to go through the New Testament. I commenced it, and read through it with attention; and thus I became aware that salvation is not merely in knowing that there is one God, and that polytheism and idolatry are false,

but that it is in the name of our most blessed Saviour, the Lord Jesus Christ; and in this manner I was cured of my second error. I afterwards read the English translation of the Koran, by Sale, the Geeta in English, and had conversations and discussions with those who knew these books in the original languages; and at last I was persuaded that what is required for man's salvation was in Christianity, and nowhere else. I then read many Christian books, together with some treatises of Hindooism and Mohamedanism, and had frequent discussions with the professors of each of them, but particularly with the latter. But the final step of baptism was difficult for me to take; for by this I was sure to lose caste and dissolve all family connection, &c.; and therefore I wished to believe that baptism and a public profession were not necessary for becoming a Christian. When I went down to Calcutta, Mrs. —— very kindly gave me a letter to the late Professor Sturt, of the Bishop's College there; and when by means of this letter I was introduced to him, though he gave me reasons for the necessity of baptism according to the Gospel, I very obstinately did not agree with him. Near the end of March, 1851, I returned to Delhi, and for more than a year I remained in great distress of mind, until the 11th of May, 1852, when I and the late Sub-Assistant Surgeon Chimmun Lall, who had formerly obtained some Christian knowledge in Calcutta, were, by God's special grace, brought to submit to baptism by the late Rev. — Jennings, chaplain of Delhi."

Ramchundra continued as teacher at Delhi College, the principal of which was Mr. F. Taylor, of whom he speaks in terms of the highest gratitude and respect. Mr. Taylor was one of the victims of the mutiny, as was also Chimmun Lall, just mentioned. "The mutineers also inquired after me; but my younger brothers, who are as yet Hindoos, concealed me in the female apartments of my family's house, in a lane, and my neighbours and acquaintances were kind enough not to betray me. On the evening of the

third day, that is, on 18th May, 1857, when it was dark, I escaped out of the city, accompanied by two faithful servants, who took me to the village of Mátola, about ten miles distant from Delhi. I remained in this village about a month, in great danger of being betrayed by those who were opposed to the zemindar who had very kindly lodged me in his house. Here I daily used to persuade the zemindars that it was wrong that the English were gone for ever, by telling them the vast resources, the power, and the knowledge of the English nation. On 10th June, 1857, a body of mutineers passed by this village, and some one told them that a Christian was living in it; but my old servant was warned of this a few minutes before: he awakened me, and told me of my danger. At first I hid myself in the zemindar's cottage, expecting to be found out and killed; but a very prudent Brahmin zemindar advised me and my servant to fly to the jungles before the mutineers could arrive. We did so; but before we could run three quarters of a mile, we heard a great noise in the village, bullets were whistling about us, and horsemen appeared to be in our pursuit, for the noise of galloping was distinctly heard. I then rushed into a thorny little bush, not minding the thorns that went into my flesh. By God's merciful providence the mutineers, after plundering and giving a good beating to the zemindars, &c. with whom I lived in the village, did not penetrate into the jungle, but went their way towards Delhi. When there was quiet towards the village, I and my old Jaut servant traversed the whole jungle, and with great difficulty reached the English camp on the 12th June, 1857. Here I was employed as an English translator of daily news from Delhi, for the information of the general and other commanders, and remained in the camp till the capture of Delhi on the 20th September, 1857. In January, 1858, I was appointed as native head master in the Thomason Civil Engineering College at Roorkee, on 250 rupees a month; which situation I held for eight months, and in the beginning of

the present month, September, 1858, I was appointed as head master of the school (not a college) which is being organized at Delhi."

Having thus given the reader the account which he will naturally expect of the reasons for this publication, and of the author of it, I leave those reasons to his attentive consideration, and that author to his kindly criticism, and to the interest which must be excited in the mind of any one who is capable of feeling curiosity about the history of human progress, by the revival in India, fostered by Europeans, of speculation on one of the sciences for which Europe is indebted to India.

A. DE MORGAN.

UNIVERSITY COLLEGE, LONDON.
January 17, 1859

A TREATISE

ON

PROBLEMS OF MAXIMA AND MINIMA,

SOLVED BY ALGEBRA.

BY RAMCHUNDRA,

TEACHER OF SCIENCE, DELHI COLLEGE

"The problems which relate to the Maxima and Minima, or the greatest or least values of variable quantities, are among the most interesting in the Mathematics, they are connected with the highest attainments of wisdom and the greatest exertions of power, and seem like so many immoveable columns erected in the infinity of space, to mark the eternal boundary which separates the regions of possibility and impossibility from one another"

2ND DISS ENCY BRIT

CALCUTTA:

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1850.

PREFACE.

For the last four or five years I was desirous of solving almost all problems of Maxima and Minima by the principles of Algebra, and not by those of the Differential Calculus. All those problems which brought out equations of the second degree were of course easily solved by the method of imaginary roots given in some works on Algebra, particularly in Wood's Algebra by Lund, and the Encyclopædia Metropolitana. But even these problems in several cases required particular artifices, without which it was impossible for me to solve them. All these problems are solved in the first chapter of this little work. Besides the method of imaginary roots, I have given another, quite independent of Imaginary quantities, quantities which to many beginners of Mathematics, appear somewhat mysterious and unintelligible. This latter method I may venture to call a *new method*, because in all mathematical works which I have had access to, I have never seen a single problem of Maxima or Minima solved by it, though it is used to reduce an adfected quadratic to a pure one in a great many works on Algebra. Thus far I have spoken of the first chapter.

All the problems solved in the second chapter bring out cubic equations, the solution of which on the condition of

Maximum or Minimum, required a new method, which I could not find, though I searched for it in several works enumerated hereafter. I then resolved to find out a method, and in intervals of leisure during three years I continually thought on the subject, and at last found it out. This is a method which appears extremely simple and easy, though it baffled all my endeavours for the space of three years. I may call it new, for I did not find it in any book I looked into.

The third and fourth chapters, and the supplement contain problems and general solutions of particular equations of the fourth, fifth, and the sixth degree, together with those problems in which two or three variable quantities enter. The methods used in these parts of the work, though more difficult and intricate than that used in the second chapter, were easily discovered.

This work contains about 130 problems taken chiefly from the following works: Simpson's Fluxions, Hall's Differential Calculus, Gregory's Examples, Connel's Differential Calculus, Walton's Differential Calculus, Ritchie's Differential Calculus, Young's Differential Calculus, Encyclopædia Britannica, Hirsch's Geometry, works on Mixed Mathematics, &c. Besides the problems solved here, many more may be solved by the methods given in this treatise.

I have also given definitions, formulæ, and propositions necessary for the study of this work in the Introduction.

In conclusion, I flatter myself with the hope that my labours will be of some use to those Mathematical students who are not advanced in their study of the Differential Calculus, and that the lovers of science, both in India and Europe, will give support to my undertaking.

Owing to the necessity of having the work printed in Calcutta, and my consequent inability to superintend the sheets passing through the press, many errors, almost inseparable from a work of this nature, have unavoidably crept in; for these I must beg the indulgence of my readers.

RAMCHUNDRA.

DELHI,
16*th February*, 1850.

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INTRODUCTION.

(1.) REDUCTIONS OF EQUATIONS.

[*Definitions.*]

1. An equation is an algebraical expression of equality between two quantities.

2. A root of an equation is that number, or quantity, which, when substituted for the unknown quantity in the equation, verifies that equation.

3. A function of a quantity is any expression involving that quantity; thus, $ax^2 + b$, $\frac{a+x}{x}$ &c. These functions are usually expressed by $f\,(x)$.

PROP. Any function of x, of the form $x^n + px^{n-1} + qx^{n-2} +$ &c., when divided by $x - a$ or $x + a$, will leave a remainder, which is the same function of a or $-a$ that the given polynomial is of x.

Let $f\,(x) = x^n + px^{n-1} + qx^{n-2} +$ &c.; and, dividing $f\,(x)$ by $x - a$ or $x + a$, let Q denote the quotient thus obtained, and R the remainder, which does not involve x; hence, by the nature of division, we have $f\,(x) = Q\,(x-a) + R$ or $f\,(x) = Q\,(x+a) + R$. Now these equations must be true for every value of x; hence, if $x = a$ the first equation becomes $f\,(a) = R$ and if $x = -a$ the second equation becomes $f\,(-a) = R$, and hence it appears that $f\,(a)$ or R is the same function of a as the given polynomial is that of x. If $f\,(x) = 0$ and a be a root of this equation, then by definition (2) we must have $f\,(a) = 0$ or $R = 0$, and hence

$\frac{f(x)}{x-a} = \frac{0}{x-a} = 0$ or $\frac{f(x)}{x+a} = \frac{0}{x+a} = 0$ or Q is in both cases $= 0$.

Ex. Let $f(x) = x^3 - x^2 + r = 0$ and $-a$ a root of this equation:

$$
\begin{array}{l}
x+a \,)\, x^3 - x^2 + r \,(\, x^2 - (a+1)\,x + a\,(a+1) \\
\qquad x^3 + ax^2 \qquad\quad = Q = 0 \ldots\ldots\ldots\ldots \text{Ans.} \\
\hline
\qquad -(a+1)\,x^2 \\
\qquad -(a+1)\,x^2 - a\,(a+1)\,x \\
\hline
\qquad\qquad a\,(a+1)\,x + r \\
\qquad\qquad a\,(a+1)\,x + a^2\,(a+1) \\
\hline
\qquad\qquad\qquad r - a^2\,(a+1) = R = 0.
\end{array}
$$

This last equation expresses the condition of a, being a negative root of the given equation.

(2.) TO FIND THE EQUATION TO THE PARABOLA. (Fig. 1.)

Let a point S be taken without the right line CB, and let the indefinite line Sm revolve about the point S in the plane SBC; also, let Cm, which is perpendicular to CB, cut Sm in m; then, if Sm be always equal to Cm, the locus of the point m is a parabola.

Through S draw BSP at right angles to CB, and if SB be bisected in A, the curve will pass through A, as appears by the construction; draw mP perpendicular to BP, and let $AP = x$, $Pm = y$, $AS = a$; then $SP^2 + Pm^2 = (Sm^2 = Cm^2) = BP^2$, or $(x-a)^2 + y^2 = (x+a)^2$; that is, $x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$, or $y^2 = 4ax$. This equation is called the equation of the Parabola, because it expresses the relation between the lines AP & Pm which determine the position of points on the curve.

(3.) TO FIND THE EQUATION TO THE ELLIPSE. (Fig. 2.)

Let two indefinite lines Sm, Hm, revolve, in a given plane, about the points S, H, and cut each other in m, in such a manner that $Sm + mH$ may be an invariable quantity; then the locus of the point m is an Ellipse. Bisect SH in C, and from m draw mP perpendicular to SH, or SH produced; let $CP = x$, $Pm = y$, $CS = c$, $Sm + Hm = 2a$. Then $\sqrt{SP^2 + Pm^2} = Sm$, and $\sqrt{HP^2 + Pm^2} = Hm$; therefore $\sqrt{SP^2 + Pm^2} + \sqrt{HP^2 + Pm^2} = Sm + Hm$, or $\sqrt{(c-x)^2 + y^2} + \sqrt{(c+x)^2 + y^2} = 2a$: hence $\sqrt{(c-x)^2 + y^2} = 2a - \sqrt{(c+x^2) + y^2}$, and squaring both sides, $c^2 - 2cx + x^2 + y^2 = 4a^2 - 4a\sqrt{(c+x)^2 + y^2} + c^2 + 2cx + x^2 + y^2$; that is, by transposition, $4a^2 + 4cx = 4a\sqrt{(c+x)^2 + y^2}$ or $a^2 + cx = a\sqrt{(c+x)^2 + y^2}$; and again squaring both sides, $a^4 + 2a^2cx + c^2x^2 = a^2c^2 + 2a^2cx + a^2x^2 + a^2y^2$, or $a^2y^2 = a^4 - a^2c^2 - (a^2 - c^2)\ x^2$; let $a^2 - c^2 = b^2$, then $a^2y^2 = a^2b^2 - b^2x^2$, and $y^2 = \frac{b^2}{a^2}\ (a^2 - x^2)$; this equation is called the equation of the Ellipse, because it expresses the relation between the lines cP and Pm, which determine the positions of points on the curve.

(4.) TO FIND THE EQUATIONS TO THE ELLIPSOID, SPHEROID, AND SPHERE. (Fig. 3.)

An Ellipsoid is a solid figure, such that sections of it perpendicular to its three axes are all Ellipses, and consequently its three axes are unequal.

A Spheroid is a solid figure, generated by the revolution of an Ellipse about its major or minor axes, and consequently two of its axes are equal to each other, and sections of it

perpendicular to the axis, about which the revolution is conceived to be performed, are all circles.

A Sphere is a solid figure, generated by the revolution of a circle about one of its diameters. Figure 3 represents the eighth part of an Ellipsoid.

AB is part of the Ellipse in the plane xy
AD .... xz
BD .... yz

And the section QPR parallel to xy is also an Ellipse.

The surface may be conceived to be generated by a variable Ellipse CAB moving upwards parallel to itself, with its centre in CZ. Let nQR be one position of this variable Ellipse; and let

$Cn = z$	$CA = a$	$nR = x_1$
$nm = x$	$CB = b$	$nQ = y_1$
$mP = y$	$CD = c$	

then from the Ellipse QPR we have

$$\frac{x^2}{x_1^2} + \frac{y^2}{y_1^2} = 1$$

Also from the Ellipses DRA and DQB we have

$$\frac{x_1^2}{a^2} + \frac{z^2}{c^2} = 1, \text{ and } \frac{y_1^2}{b^2} + \frac{z^2}{c^2} = 1$$

therefore $\frac{x_1^2}{a^2} = \frac{y_1^2}{b^2}$; and, multiplying the first equation by $\frac{x_1^2}{a^2}$ or its equal $\frac{y_1^2}{b^2}$, we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x_1^2}{a^2} = 1 - \frac{z^2}{c^2}$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ equation to the Ellipsoid;}$$

let $a = b \quad \therefore \quad \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$ equation to the Spheroid;

let $a = b = c \therefore \frac{x^2 + y^2 + z^2}{a^2} = 1$ equation to the Sphere.

(5.) TO FIND THE AREA OF A TRIANGLE. (Fig. 4.)

Rule 1st.—Multiply the base by the perpendicular height, and half the product will be the area. The truth of this rule is evident, because any triangle is the half a parallelogram of equal base and altitude, by Euclid, prop. 41, 1st Book.

Rule 2nd.—When the three sides are given: add all the three sides together, and take half that sum. Next, subtract each side severally from the said half sum, obtaining three remainders.

Lastly, multiply the said half sum and those three remainders all together, and extract the square root of the last product, for the area of the triangle. For let $a,b,c,$ denote the sides opposite respectively to $A,B,C,$ the angles of the triangle ABC; then by prop. 13, of Euclid, book 1st, we have $BC^2 = AB^2 + AC^2 - 2AB.\ AP$, or $a^2 = b^2 + c^2 - 2c.\ AP$

or $AP = \dfrac{b^2 + c^2 - a^2}{2c}$ hence we have

$$CP^2 = b^2 - \frac{(b^2 + c^2 - a^2)^2}{4c^2} = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4c^2}$$

$$= \frac{(2bc + b^2 + c^2 - a^2)\ (2bc - b^2 - c^2 + a^2)}{4c^2}$$

$$\therefore 4c^2\, CP^2 = \left\{ (b + c)^2 - a^2 \right\} \left\{ a^2 - (c - b)^2 \right\}$$

$$= (a + b + c)\ (-a + b + c)\ (a - b + c)\ (a + b - c)$$

$$\therefore \tfrac{1}{2}AB.CP = \tfrac{1}{2}c.CP = \sqrt{\left\{\frac{a + b + c}{2} \cdot \frac{-a + b + c}{2} \cdot \frac{a - b + c}{2} \cdot \frac{a + b - c}{2}\right\}} = \sqrt{s\,(s - a)\ (s - b)\ (s - c)}$$

where $s = \frac{1}{2}\ (a + b + c)$ = half the sum of the three sides.

(6.) TO FIND THE DIAMETER AND CIRCUMFERENCE OF ANY CIRCLE, THE ONE FROM THE OTHER. (Fig. 5.)

This may be done by the following proportion, viz. As 1 is to 3·1416, so is the diameter to the circumference. For, let *ABCD* be any circle, whose centre is *E*, and let *AB*, *BC*, be any two equal arcs. Draw the several chords as in the figure, and join *BE*; also draw the diameter *DA*, which produce to *F*, till *BF* be equal to the chord *BD*. Then the two isosceles triangles *DEB*, *DBF*, are equiangular, because they have the angle at *D* common; consequently $DE : DB :: DB : DF$. But the two triangles *AFB*, *DCB*, are identical, or equal in all respects, because they have the angle $F =$ the angle *BDC*, being each equal to the angle *ADB*, these being subtended by the equal arcs *AB*, *BC*; also the exterior angle *FAB* of the quadrangle *ABCD*, is equal to the opposite interior angle at *C*; and the two triangles have also the side $BF =$ side *BD*; therefore the side *AF* is also equal to the side *DC*. Hence the proportion above, viz. $DE : DB :: DB : DF = DA + AF$ becomes $DE : DB :: DB : 2DE + DC$. Then by taking the rectangles of the extremes and means, it is $DB^2 = 2DE^2 + DE . DC$. Now if the radius $DE = 1$, this expression becomes $DB^2 = 2 + DC$, and hence $DB = \sqrt{2 + DC}$. That is, if the measure of the supplemental chord of any arc be increased by the number 2, the square root of the sum will be the supplemental chord of half that arc. Let $AC =$ a side of the inscribed regular hexagon $= 1 \therefore DC = \sqrt{AD^2 - AC^2} = \sqrt{2^2 - 1} = \sqrt{3} = 1·7320508076$, the supplemental chord of $\frac{1}{6}$ of the periphery. Then, by the foregoing theorem, by always bisecting the arcs, and adding 2 to the last square root, there will be found the supplemental chords of the 12th, the 24th the 48th, 96th, &c., to the 1536th part of the periphery; thus

it is found that 3·9999832669 is the square of the supplemental chord of the 1536th part of the periphery; let this number be taken from 4, the square of the diameter, and the remainder = 0·0000167331 $\therefore \sqrt{0{\cdot}0000167331} =$ 0·0040906112 = $\frac{1}{1536}$ of the periphery; this number then being multiplied by 1536, gives 6·2831788 for the perimeter of a regular polygon of 1536 sides inscribed in the circle = the circumference very nearly when the diameter of the circle = 2.

(7.) THE AREA OF ANY CIRCLE = RECTANGLE OF ½ CIRCUMFERENCE AND ½ ITS DIAMETER. (Fig. 6.)

Conceive a regular polygon to be inscribed in a circle; and radii drawn to all the angular points, dividing it into as many equal triangles as the polygon has sides, one of which is *ABC*, of which the altitude is the perpendicular *CD* from the centre to the base *AB*.

Then the triangle *ABC* is equal to a rectangle of half the base *AD* and the altitude *CD*; consequently, the whole polygon, or all the triangles added together which compose it, is equal to the rectangle of the common altitude *CD*, and the halves of all the sides, or the half perimeter of the polygon.

Now, conceive the number of sides of the polygon to be indefinitely increased; then will its perimeter coincide with the circumference of the circle, and consequently the altitude *CD* will become equal to the radius, and the whole polygon equal to the circle. Consequently, the space of the circle, or of the polygon in that state, is equal to the rectangle of the radius and half the circumference. *Q.E.D.*

(8.) EVERY SPHERE IS TWO-THIRDS OF ITS CIRCUMSCRIBING CYLINDER. (Fig. 7.)

By prop. 12 of Euclid, Book 12th, the cones AIB and QIM are in the triplicate ratio of IF and IK, that is to say we have this proportion—

Cone AIB : cone QIM :: $IF^3 : IK^3 :: FH^3 : (FH - 2FK)^3$

$\therefore$ Cone AIB : frustum $ABMQ$ :: $FH^3 : FH^3 - (FH - 2FK)^3$

$:: FH^3 : 6FH^2FK - 12FHFK^2 + 8FK^3$ but cone AIB = one-third of the cylinder $ABGE$, hence;

Cylinder AG : frustum $ABMQ$ :: $3FH^3 : 6FH^2.FK - 12FH.FK^2 + 8FK^3$.

Now cylinder AL : cylinder AG :: $FK : FI$.

$\therefore$ Cylinder AL : $ABMQ$ :: $6FH^2 : 6FH^2 - 12FH.FK + 8FK^2$, ... (1)

Now it is evident that $IK = KM \therefore IK^2 + KN^2 = KM^2 + KN^2 = IN^2 = IG^2 = KL^2$. Now circles are to each other as the squares of their diameters, or of their radii; therefore the circle described by KL is equal to both the circles described by KM and KN; or the section of the cylinder is equal to both the corresponding sections of the sphere and cone. And as this is always the case in every parallel position of KL, it follows that the cylinder EB, which is composed of all the former sections, is equal to the hemisphere EFG and cone IAB, which are composed of all the latter sections. By proportion (1) we find

Cylinder AL : segment PFN :: $6FH^2 : 12FH.FK - 8FK^2$ div.

$$:: \tfrac{3}{2}FH^2 : FK\,(3FH - 2FK)$$

But cylinder AL = circular base, whose diameter is AB or FH multiplied by the height FK; hence cylinder AL = circle $EFGH \times FK$.

$\therefore$ Segment $PFN = \frac{2}{3}$. circle $\frac{EFGH}{FH^2}$ $(3FH - 2FK)$ FK^2 .... (2)

If $FK = FH$, then the sphere $= \frac{2}{3}$ cylinder. *Q.E.D.*

NOTE.—For the cylinder AL = frustrum $ABMQ$ + segment PFN and $\therefore$ cylinder AL − frustrum $ABMQ$ = segment PFN.

(9.) TO FIND THE AREA OF AN ELLIPSE. (Fig. 8.)

The equation to the Ellipse is $y = \frac{b}{a}\sqrt{a^2 - x^2}$ and to the circle described on the major axis as diameter is $y^1 = \sqrt{a^2 - x^2}$. Comparing these two equations we find

$y = \frac{b}{a}y^1$ or $2\,a\,y = 2by^1$, and $\therefore y : y^1 :: 2b : 2a$.

In the diagram annexed $2a = A^1A$, $2b = B^1B$, $x =$ any of the lines or abscissas measured on the line CA or CA^1 from the point C, $y =$ any of the perpendicular lines denoted by pm or p^1m which are called the ordinates of the Ellipse, and $y^1 =$ any of the perpendicular lines denoted by Pm or P^1m which are called the ordinates of the Circle. Now if the area of the Ellipse and Circle be supposed to be divided into bands perpendicular to the axis major AA^1, by ordinates Ppm, placed so closely together that the arcs of the curves between them may be considered to be straight lines, the areas of the spaces of the Ellipse and Circle between every pair of contiguous ordinates will be proportional to those ordinates, and as all the ordinates are in the same ratio, the sum of all the areas between the elliptical ordinates, that is, the area of the Ellipse itself, will be to the sum of all the areas included between the circular ordinates, that is, to the area of the Circle itself, as any elliptical ordinate is to the corresponding circular ordinate, that is, as the axis minor of the Ellipse is to its axis major,

By article 6th we find that the circumference of the Circle described upon the major axis is to its diameter as 2 is to 6·2831 &c. or 1 : 3·1415 &c. (which let$=p$) :: $2a$: circumference $= 2pa$ $\therefore$ $\frac{1}{2}$ circum. $= pa$ and a = semi-diameter $\therefore$ the area of the Circle $= pa \times a = pa^2$, we therefore find area of the Ellipse : pa^2 :: $2b$: $2a$ $\therefore$ area of the Ellipse $= pab$.

(10.) TO FIND THE SUM OF n TERMS OF THE SERIES
$1+4+9+16+25+\ldots\ldots n^2$.

Assume $1 + 4 + 9 + 16 + 25 + \ldots\ldots n^2 = Pn^3 + Qn^2 + Rn + S$, and since there are four co-efficients to be determined, we must have a corresponding number of independent equations; hence

when $n=1$ we have $P+Q+R+S=1$

$n=2$ ...... $8P+4Q+2R+S=1+4=5$

$n=3$ ...... $27P+9Q+3R+S=1+4+9=14$

$n=4$ ...... $64P+16Q+4R+S=1+4+9+16=30$.

And from these four simple equations we find, by continued subtraction, $P=\frac{1}{3}$, $Q=\frac{1}{2}$, $R=\frac{1}{6}$ and $S=0$; therefore the sum of $1 + 4 + 9 + 16 + 25 + \ldots\ldots n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n}{6}(2n^2+3n+1) = \frac{n(n+1)}{2}\,\frac{(2n+1)}{3}$. If n be supposed to be indefinitely great, n and $2n$ may be put instead of $(n+1)$ and $(2n+1)$ and $\therefore$ in this case the sum of the series $= \frac{n^3}{3}$. .... (A.)

(11.) TO FIND THE AREA OF A PARABOLA. (Fig. 9.)

The equation to the parabola is $y^2 = 4ax$ and consequently we have the following equations:—

$$Kp^2 = 4aAK \;\therefore\; AH^2 = 4aHp \text{ or } Hp = \frac{AH^2}{4a}$$

$$Ln^2 = 4aAL \;\therefore\; AG^2 = 4aGn \text{ or } Gn = \frac{AG^2}{4a} \;\&c. = \&c.$$

$$AF^2 = 4aFr \text{ or } Fr = \frac{AF^2}{4a} \;\&c. = \&c. = \&c.$$

Let $AH = HG = GF = FD = \&c.$ and each $= \frac{AD}{n}$

$$\therefore \text{rect. } HK = Hp \times AH = \frac{AH^2}{4a} \times AH = \frac{AH^3}{4a} = \frac{AD^3}{4an^3}$$

$$\text{rect. } Gq = HG \times Gn = HG \times \frac{4HG^2}{4a} = \frac{4AD^3}{4an^3}$$

$$\text{rect. } Fw = GF \times Fr = GF \times \frac{9GF^2}{4a} = \frac{9AD^3}{4an^3} \;\&c. = \&c.$$

$$\therefore \text{The sum of these rectangles} = \frac{AD^3}{4an^3} + \frac{4AD^3}{4an^3} + \frac{9AD^3}{4an^3} + \&c.$$

$$= \frac{AD^3}{4an^3}(1 + 4 + 9 + \ldots n^2) = \frac{AD^3}{4an^2}\,\frac{(n+1)}{2.}\,\frac{(2n+1)}{3.} =$$

$$\frac{AD^2}{4a} \times \frac{AD}{n^2}\,\frac{(n+1)}{2.}\,\frac{(2n+1)}{3} = DC \times \frac{AD}{n^2}\,\frac{(n+1)}{2.}\,\frac{(2n+1)}{3}$$

It is evident that if the number of parts into which the line AD is divided be infinitely great, the sum of the rectangles must be equal to the area $ApnrCD$ and also by art. 10, equa. (A) $\frac{(n+1)}{2.}\,\frac{(2n+1)}{3} = \frac{n^2}{3}$ $\therefore$ the area $Apnr\ CD$

$$\frac{DC \times AD}{n^3} \times \frac{n^2}{3} = \frac{DC \times AD}{3} \;\therefore \text{ the area } Apnr\ CD \text{ of the}$$

parabola $=$ rect. $AD \times AB - Apnr\ CD = DC \times AD - \frac{DC \times AD}{3}$ $\frac{2DC \times AD}{3}$ Q.E.D.

A TREATISE ON PROBLEMS OF MAXIMA AND MINIMA SOLVED BY ALGEBRA.

CHAPTER I.

Problems in the solution of which simple and quadratic equations only are used.

PROB. (1.) TO DIVIDE A GIVEN NUMBER INTO TWO SUCH PARTS THAT THEIR PRODUCT MAY BE THE GREATEST POSSIBLE.

Put the given number $= a$, one of the parts required $= x$, and consequently $a - x =$ the other part, $\therefore x(a - x) = ax - x^2 =$ product $=$ maximum, which let $= r \therefore x^2 - ax = -r$. Solving this quadratic equation we find $x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - r}$. Now it is evident that r cannot be greater than $\frac{a^2}{4}$ for if it be so, the value of x becomes impossible; therefore the product $ax - x^2$ or r is greatest when $\frac{a^2}{4} = r$, and $\therefore x = \frac{a}{2}$.

The same solved without impossible roots.

In the expression $ax - x^2$ which is to become a maximum, let $x = y + \frac{a}{2}$ where the value of y determined by the condition of $ax - x^2$ being a maximum, will show whether it is positive, zero, or negative. We now find

$ax - x^2 = ay + \frac{a^2}{2} - y^2 - ay - \frac{a^2}{4} = \frac{a^2}{4} - y^2$, which is evidently a maximum when $y = 0$, $\therefore x = \frac{a}{2}$ as before.

PROB. (2.) TO DETERMINE THE GREATEST RECTANGLE INSCRIBED IN A GIVEN TRIANGLE. (Fig. 10.)

Let the base AC of the given triangle $= b$, and its altitude $BD = a$, and let the altitude BS of the inscribed rectangle mc (considered as variable) be denoted by x, Then, because of the parallel lines AC, ac, we find the proportion,

$BD : AC :: DS : ac$ or $a : b :: a - x : ac$ or $ac = \dfrac{ab - bx}{a}$; whence the area of the rectangle or $ac \times BS = \dfrac{bax - bx^2}{a} = \dfrac{b}{a}(ax - x^2) =$ max. It is evident that when a quantity is a maximum, any determinate part, multiple or power of it must also be a maximum, and consequently the determinate $ax - x^2$ of $\dfrac{b}{a}(ax - x^2)$ must be $=$ max. which let $= r \therefore ax - x^2 = r$ or $x^2 - ax = -r$. Solving this quadratic equation we find,

$x = \dfrac{a}{2} \pm \sqrt{\dfrac{a^2}{4} - r}$, and it is manifest now that $ax - x^2$ or r cannot be greater than $\dfrac{a^2}{4}$ (for the reason stated in the last problem), and, therefore, when $r =$ max. we must have $r = \dfrac{a^2}{4} \therefore x = \dfrac{a}{2}$. Whence the greatest inscribed rectangle is that whose altitude is just half the altitude of the triangle.

The same solved without impossible roots.

In the expression $ax - x^2$, which is to become a maximum, let $x = y + \dfrac{a}{2} \therefore ax - x^2 = a\left(y + \dfrac{a}{2}\right) - \left(y + \dfrac{a}{2}\right)^2 = ay + \dfrac{a^2}{2} - y^2 - ay - \dfrac{a^2}{4} = \dfrac{a^2}{4} - y^2$, which is evidently $=$ max. when $y = 0$, or $x = \dfrac{a}{2}$ as before.

PROB. (3.) OF ALL RIGHT-ANGLED PLANE TRIANGLES HAVING THE SAME GIVEN HYPOTHENUSE, TO FIND THAT (ABC) WHOSE AREA IS THE GREATEST POSSIBLE. (Fig. 11.)

Let $AC = a$, $AB = x$ and $BC = y$. Then, $x^2 + y^2$ being $= a^2$ we shall have $y = \sqrt{a^2 - x^2}$, and consequently $\frac{xy}{2} = \frac{x}{2}\sqrt{a^2 - x^2} =$ the area of the triangle $=$ max. and consequently the square of the area, or $\frac{a^2x^2 - x^4}{4} =$ max. and also four times this, or $a^2x^2 - x^4 =$ max. which let $= r$. $\therefore x^4 - a^2x^2 = -r$. Solving this quadratic equation we find $x^2 = \frac{a^2}{2} \pm \sqrt{\frac{a^4}{4} - r}$, and it is manifest that $a^2x^2 - x^4$ or r cannot be greater than $\frac{a^4}{4}$; therefore when $r =$ max. we must have $r = \frac{a^4}{4}$, $\therefore x^2 = \frac{a^2}{2}$ and $x = \frac{a}{\sqrt{2}}$, and $y = \sqrt{a^2 - x^2}$ $= \frac{a}{\sqrt{2}}$. Hence it appears that the right-angled plane triangle contains the greatest area whose two sides containing the right angle are equal to each other.

The same solved without impossible roots.

In the expression $a^2x^2 - x^4$ which is to become maximum let $x^2 = y^2 + \frac{a^2}{2}$ $\therefore a^2x^2 - x^4 = a^2\left(y^2 + \frac{a^2}{2}\right) - \left(y^2 + \frac{a^2}{2}\right)^2$ $= a^2y^2 + \frac{a^4}{2} - y^4 - a^2y^2 - \frac{a^4}{4} = \frac{a^4}{4} - y^4$, which is evidently $=$ maximum when $y^4 = 0$, and $\therefore x^2 = \frac{a^2}{2}$ or $x = \frac{a}{\sqrt{2}}$ as before.

PROB. (4.) OF ALL RIGHT-ANGLED PLANE TRIANGLES CONTAINING THE SAME GIVEN AREA, TO FIND THAT WHEREOF THE SUM OF THE TWO SIDES, $AB+BC$, IS THE LEAST POSSIBLE. (See Fig. 11.)

Let one leg AB, be denoted by x, and the area of the triangle by a; then the other side will be denoted by $\frac{2a}{x}$, and the sum of the two legs will be $x + \frac{2a}{x}$ = minimum, which let $= r \therefore x^2 - rx = -2a$ ....(1.)

Solving this quadratic equation we find $x = \frac{r}{2} \pm \sqrt{\frac{r^2}{4} - 2a}$, and it is evident now that r cannot be so small as to make $\frac{r^2}{4}$ less than $2a$; therefore, when $r =$ min. we must have $\frac{r^2}{4} = 2a \therefore r = 2\sqrt{2a}$ and $x = \frac{r}{2} = \sqrt{2a} = AB$. Whence $BC = \frac{2a}{x}$ is also $= \sqrt{2a}$. Therefore the two legs are equal to each other.

The same solved without impossible roots.

From equation (1) in the preceding solution we have $x^2 - rx = -2a$, and $\therefore rx - x^2 = 2a$. Let $x = y + \frac{r}{2} \therefore rx - x^2 = r\left(y + \frac{r}{2}\right) - \left(y + \frac{r}{2}\right)^2 = ry + \frac{r^2}{2} - y^2 - ry - \frac{r^2}{4} = 2a$, or $\frac{r^2}{4} - y^2 = 2a, \therefore r^2 = 8a + 4y^2$. Now it is evident r or r^2 is the least possible when $y = 0$, $\therefore r = 2\sqrt{2a}$ and $x = \sqrt{2a}$ as before.

PROB. (5.) DIVIDE A GIVEN LINE, AB, INTO TWO PARTS, SO THAT THE SUM OF THE AREAS OF THE SQUARES DESCRIBED ON THESE PARTS SHALL BE THE LEAST POSSIBLE.

Let $a =$ the given line, x one of the parts, then $a - x$ will be the other part. Then, $x^2 + (a-x)^2$ is a minimum, that is $2x^2 + a^2 - 2ax$ is a minimum. Now a^2 is a given determinate quantity and therefore when $2x^2 + a^2 - 2ax =$ minimum we must also have $2x^2 - 2ax =$ minimum or its half, viz. $x^2 - ax =$ minimum, which let $= r \therefore x^2 - ax = r$. Solving this quadratic equation we find $x = \frac{a}{2} \pm \sqrt{r + \frac{a^2}{4}}$ Now r can be less than zero, that is it may become negative; but, when negative, it cannot be so great as to make the radical impossible. Therefore, when the least possible, r must become a negative quantity $= -\frac{a^2}{4}$ and hence $x = \frac{a}{2}$. This problem may be solved by the following method which is more elegant.

Let $2x^2 + a^2 - 2ax = r$, $\therefore x^2 - ax = \frac{r}{2} - \frac{a^2}{2}$ .... (1)

Solving this quadratic equation we find,

$x = \frac{a}{2} \pm \sqrt{\frac{r}{2} - \frac{a^2}{4}}$. Now r or $\frac{r}{2}$ cannot be so small as to make $\frac{r}{2}$ less than $\frac{a^2}{4}$, because in this case the radical quantity becomes impossible; therefore when r is the least possible, we must have $\frac{r}{2} = \frac{a^2}{4}$ and $\therefore x = \frac{a}{2}$ Hence the given line must be bisected.

The same solved without impossible roots.

In the equation (1) let $x = y + \frac{a}{2}$ and therefore we find

$\left(y+\frac{a}{2}\right)^2 - a\left(y+\frac{a}{2}\right) = \frac{r}{2} - \frac{a^2}{2}$ or $y^2 + \frac{a^2}{4} = \frac{r}{2}$, and therefore $r = 2y^2 + \frac{a^2}{2}$. Now it is evident that r is the least possible when y or $2y^2 = 0$, $\therefore x = \frac{a}{2}$ as before.

* It must here be remarked that when in the solution of problems of minima we leave out some given negative quantity, we sometimes make r, or the minimum quantity, less than zero or negative, as is done in the first method of solution of the preceding problem.

PROB. (6.) OF ALL CONES UNDER THE SAME GIVEN SUPERFICIES (s) TO FIND THAT ABD WHOSE SOLIDITY IS THE GREATEST. (Fig. 12.)

Let the radius of the base $AC = x$, and the length of the slant side $AB = y$, and let p denote the periphery (3·14 &c.) of the circle whose diameter is unity. Then the circumference of the base will be $2px$, the area of the base $= px^2$, and the convex superficies of the cone $= pxy$ (which last is found by multiplying half the periphery of the base by the length of the slant side). Wherefore, since the whole superficies is $= px^2 + pxy = s$, we have $y = \frac{s}{px} - x$; whence the altitude $CB = \sqrt{\overline{AB}^2 - \overline{AC}^2}$ $\sqrt{\frac{s^2}{p^2x^2} - \frac{2s}{p}}$; which multiplied by $\frac{px^2}{3}$, or $\frac{1}{3}$ of the area of the base, gives $\frac{px^2}{3}\sqrt{\frac{s^2}{p^2x^2} - \frac{2s}{p}}$ for the solid contents of the cone; which being a maximum, its square $\frac{1}{9}(s^2x^2 - 2psx^4) = \frac{2ps}{9}$

$\left(\frac{s}{2p}x^2 - x^4\right)$ must also be a maximum. Since $\frac{2ps}{9}$ is a constant given quantity, therefore $\frac{s}{2p}x^2 - x^4$ must also be = maximum, which let $= r$ $\therefore$ $\frac{s}{2p}x^2 - x^4 = r$, and $x^4 - \frac{s}{2p}x^2 = -r$. Solving this quadratic equation we find $x^2 = \frac{s}{4p} \pm \sqrt{\frac{s^2}{16p^2} - r}$, $\therefore$ when $r =$ max. it must be $= \frac{s^2}{16p^2}$, $\therefore$ $x^2 = \frac{s}{4p}$ and $x = \sqrt{\frac{s}{4p}}$. Now $y = \frac{s}{px} - x = \frac{s}{p\sqrt{\frac{s}{4p}}} - \sqrt{\frac{s}{4p}} = \frac{\sqrt{4ps}}{p} - \sqrt{\frac{s}{4p}} = \frac{4\sqrt{s} - \sqrt{s}}{\sqrt{4p}} = \frac{3\sqrt{s}}{2\sqrt{p}}$. Hence it appears that the greatest cone under a given surface (or a given cone under the least surface) will be, when the length of the slant side is to the semi-diameter of the base in the ratio of 3 to 1, or (which comes to the same thing) when the square of the altitude is to that of the whole diameter in the ratio of 2 to 1.

The same solved without impossible roots.

In the expression $\frac{s}{2p}x^2 - x^4 =$ max. let $x^2 = \frac{s}{4p} + y$ $\therefore$ $\frac{s}{2p}x^2 - x^4 = \frac{s^2}{8p^2} + \frac{s}{2p}y - \left(\frac{s}{4p} + y\right)^2 = \frac{s^2}{8p^2} + \frac{s}{2p}y - \frac{s^2}{16p^2} - \frac{s}{2p}y - y^2 = \frac{s^2}{16p^2} - y^2 =$ max. when $y = 0$, $\therefore$ $x^2 = \frac{s}{4p}$ and $x = \sqrt{\frac{s}{4p}}$ as before.

PROB. (7.) TO DETERMINE THE POSITION OF THE RIGHT LINE DE, WHICH, PASSING THROUGH A GIVEN POINT P SHALL CUT TWO RIGHT LINES AR AND AS, GIVEN IN POSITION, IN SUCH SORT THAT THE SUM OF THE SEGMENTS, AD AND AE, MADE THEREBY, MAY BE THE LEAST POSSIBLE. (Fig. 13.)

Make PB, parallel to AS, $= a$, and PC, parallel to AR, $= b$; and let $BD = x$. Then by reason of the parallel lines, we will have the proportion $x : a :: b : CE = \frac{ab}{x}$: Therefore $AD + AE = b + a + x + \frac{ab}{x} =$ minimum. Now $b + a$, being a constant given quantity, $x + \frac{ab}{x}$ is also a minimum, which let $= r$, $\therefore x + \frac{ab}{x} = r$ or $x^2 - rx = -ab$. Solving this quadratic we find $x = \frac{r}{2} \pm \sqrt{\frac{r^2}{4} - ab}$ or $r = 2\sqrt{ab}$ and $x = \frac{r}{2} = \sqrt{ab}$.

The same solved without impossible roots.

Since $x^2 - rx = -ab$, we find $rx - x^2 = ab$. Let $x = y + \frac{r}{2} \therefore rx - x^2 = ry + \frac{r^2}{2} - \left(y + \frac{r}{2}\right)^2 = ry + \frac{r^2}{2} - y^2 - ry - \frac{r^2}{4} = \frac{r^2}{4} - y^2 = ab$ or $r^2 = 4ab + 4y^2 =$ min. when $y = 0$ and therefore $r = 2\sqrt{ab}$ and $x = \frac{r}{2} = \sqrt{ab}$ as before.

PROB. (8.) IF TWO BODIES MOVE AT THE SAME TIME, FROM TWO GIVEN PLACES A AND B, AND PROCEED UNIFORMLY FROM THENCE IN GIVEN DIRECTIONS, AP AND BQ, WITH CELERITIES IN A GIVEN RATIO, IT IS PROPOSED TO FIND THEIR POSITION, AND HOW FAR EACH HAS GONE, WHEN THEY ARE THE NEAREST POSSIBLE TO EACH OTHER. (Fig. 14.)

Let M and N be two cotemporary positions of the bodies, and upon AP let fall the perpendiculars NE and BD; also let QB be produced to meet AP in C, and let MN be drawn: moreover, let the given celerity in BQ be to that in AP, as n is to m, and let AC, BC, and CD (which are also given), be denoted by a, b, and c, respectively, and make the variable distance $CN=x$: Then, by reason of the parallel lines NE and BD, we shall have $CB : CN :: CD : CE$ or $b : x :: c : CE \therefore CE = \frac{cx}{b}$. Also, because the distances, BN and AM, gone over in the same time, are as the celerities, we likewise have, $n : m :: BN : AM$ or $n : m :: x - b : AM$, or $AM = \frac{mx - mb}{n}$, and consequently $CM = (AC - AM) = a + \frac{mb}{n} - \frac{mx}{n} = d - \frac{mx}{n}$, (by writing $d = a + \frac{mb}{n}$). Whence $MN^2 = CM^2 + CN^2 - CM \times 2CE = \left(d - \frac{mx}{n}\right)^2 + x^2 - \left(d - \frac{mx}{n}\right) \times \frac{2cx}{b} = d^2 - \frac{2dmx}{n} + \frac{m^2x^2}{n^2} + x^2 - \frac{2cdx}{b} + \frac{2cmx^2}{nb} = \left(\frac{m^2}{n^2} + 1 + \frac{2cm}{nb}\right)x^2 - \left(\frac{2dm}{n} + \frac{2cd}{b}\right)x + d^2 = \left(\frac{m^2}{n^2} + 1 + \frac{2cm}{nb}\right)$

$$\left\{x^2 - \left(\frac{\frac{2dm}{n} + \frac{2cd}{b}}{\frac{m^2}{n^2} + 1 + \frac{2cm}{nb}}\right)x + \frac{d^2}{\frac{m^2}{n^2} + 1 + \frac{2cm}{nb}}\right\}$$

Now let the quantity without the brackets $= Q$, the co-efficient of $x = A$ and $\dfrac{d^2}{\frac{m^2}{n^2} + 1 + \frac{2cm}{nb}} = B$, and we shall therefore find $Q\,(x^2 - Ax + B) =$ minimum or $x^2 - Ax + B$ min. which let $= r$, and $\therefore x^2 - Ax + B = r$ or $x^2 - Ax = r - B$ ....(1.)

Before solving this equation we must show that $\frac{A^2}{4}$ is less than B. Since $c = CD$ and $b = BC \therefore b > c$ and $b^2 > c^2$ $\therefore n^2b^2 > n^2c^2$ ....(2.)

Now $A = \dfrac{\frac{2dm}{n} + \frac{2cd}{b}}{\frac{m^2}{n^2} + 1 + \frac{2cm}{nb}} = \dfrac{2nd\,(bm + cn)}{m^2b + n^2b + 2mnc} \therefore \dfrac{A^2}{4}$ $= \dfrac{n^2d^2\,(bm + cn)^2}{(m^2b + n^2b + 2mnc)^2}$ and $B = \dfrac{bd^2n^2}{m^2b + n^2b + 2mnc} = \dfrac{n^2d^2\,(m^2b^2 + n^2b^2 + 2mnbc)}{(m^2b + n^2b + 2mnc)^2}$. We therefore find $B : \frac{A^2}{4} ::$ $m^2b^2 + n^2b^2 + 2mnbc : m^2b^2 + 2mnbc + c^2n^2$. Now as $m^2b^2 = m^2b^2$, $2mnbc = 2mnbc$, and $n^2b^2 > n^2c^2$ by inequation (2) $\therefore$ the third term of this proportion is greater than the fourth $\therefore B$ is greater than $\frac{A^2}{4}$ and $\therefore \frac{A^2}{4} - B =$ a negative quantity, and may therefore be supposed $= -P$. The equation (1) gives $x^2 - Ax = r - B \therefore x = \frac{A}{2} \pm \sqrt{r + \frac{A^2}{4} - B}$ $= \frac{A}{2} \pm \sqrt{r - P}$. Now r cannot be less than $P \therefore r =$ min. when $r = P \therefore x = \dfrac{A}{2} = \dfrac{mnbd + n^2cd}{m^2b + n^2b + 2mnc}$; from whence BN, AM, and MN are also given.

The same solved without impossible roots.

In the expression $x^2 - Ax + B = \text{min.}$ let $x = y + \frac{A}{2}$ $\therefore x^2 - Ax + B = \left(y + \frac{A}{2}\right)^2 - A\left(y + \frac{A}{2}\right) + B = y^2 + Ay + \frac{A^2}{4} - Ay - \frac{A^2}{2} + B = y^2 + B - \frac{A^2}{4}$; but $\frac{A^2}{4} - B = -P \therefore B - \frac{A^2}{4} = P \therefore y^2 + P = \text{min.}$ which is the case when $y = 0 \therefore x = \frac{A}{2} = \frac{mnbd + n^2cd}{m^2b + n^2b + 2mnc}$ as before.

PROB. (9.) LET THE BODY M MOVE UNIFORMLY, FROM A TOWARDS Q, WITH THE CELERITY m, AND LET ANOTHER BODY N PROCEED FROM B, AT THE SAME TIME, WITH THE CELERITY n. NOW IT IS PROPOSED TO FIND THE DIRECTION BD OF THE LATTER, SO THAT THE DISTANCE MN OF THE TWO BODIES WHEN THE LATTER ARRIVES IN THE WAY OF DIRECTION AQ OF THE FORMER, MAY BE THE GREATEST POSSIBLE. (Fig. 14.)

Let BC be perpendicular to AQ, and make $AC = a$, $BC = b$, and $BN = x$. Therefore, if the position M, be supposed cotemporary with N, we shall have

$n : m :: x : AM \therefore AM = \frac{mx}{n}$; whence $CM = \frac{mx}{n} - a$, and consequently $MN = (CN - CM) = \sqrt{x^2 - b^2} - \frac{mx}{n} + a =$ max. which let $= r, \therefore \sqrt{x^2 - b^2} = r + \frac{mx}{n} - a$, and $\therefore x^2 - b^2 = \left(r + \frac{mx}{n} - a\right)^2 = r^2 + \frac{2rmx}{n} + \frac{m^2x^2}{n^2} - 2ar - \frac{2amx}{n} + a^2$, $\therefore 2ar - r^2 - b^2 - a^2 = \frac{m^2 - n^2}{n^2}x^2 - \frac{2m(a-r)}{n}x$

$\therefore x^2 - \frac{2mn(a-r)}{m^2-n^2}x = \frac{(2ar - r^2 - b^2 - a^2)n^2}{m^2 - n^2}$ ...... (1) and

$x^2 - \frac{2mn(a-r)}{m^2-n^2}x + \frac{m^2n^2(a-r)^2}{(m^2-n^2)^2} = \frac{m^2n^2(a-r)^2}{(m^2-n^2)^2} +$
$\frac{(2ar - r^2 - b^2 - a^2)n^2(m^2 - n^2)}{(m^2-n^2)^2}$ and therefore $x = \frac{mn(a-r)}{m^2-n^2}$
$\pm \sqrt{\frac{n^4(a-r)^2 - n^2b^2(m^2-n^2)}{(m^2-n^2)^2}}$.

Now it is evident that in order that this problem may be possible, r must be less* than a, and consequently $r = \max.$ when $n^4(a-r)^2 = n^2b^2(m^2-n^2)$, for r cannot be taken so great as to render $n^4(a-r)^2 > n^2b^2(m^2-n^2)$, and therefore $a - r = \frac{b\sqrt{m^2-n^2}}{n}$ and $x = \frac{mn(a-r)}{m^2-n^2} = \frac{mb}{\sqrt{m^2-n^2}}$, and $CN = \sqrt{x^2 - b^2} = \frac{nb}{\sqrt{m^2-n^2}}$: Whence $m : n :: BN : CN :$ Radius : cosine N.

It is also evident that this problem is impossible when $m < n$.

The same solved without impossible roots.

In the equation (1) let the coefficient of $x = A$ and the second member $= B \therefore x^2 - Ax = B$. Now let $x = y + \frac{A}{2} \therefore x^2 - Ax = \left(y + \frac{A}{2}\right)^2 - A\left(y + \frac{A}{2}\right) = y^2 + Ay + \frac{A^2}{4} - Ay - \frac{A^2}{2} = y^2 - \frac{A^2}{4} = B$, and therefore $y^2 = B + \frac{A^2}{4} = \frac{n^4(a-r)^2 - n^2b^2(m^2-n^2)}{(m^2-n^2)^2}$ by substitution, $\therefore (a-r)^2 = \frac{y^2(m^2-n^2)^2 + n^2b^2(m^2-n^2)}{n^4}$, and therefore $a - r =$

* This is evident, because if $r = a$ the root becomes impossible, and if $r > a$, there can be no limit to its increase. that is, it cannot admit of being a maximum.

$\sqrt{\dfrac{y^2(m^2-n^2)^2+n^2b^2(m^2-n^2)}{n^4}}$ or $r = a -$ $\sqrt{\dfrac{y^2(m^2-n^2)^2+n^2b^2(m^2-n^2)}{n^4}}$. Now it is evident that $r =$ max. when the quantity subtracted from $a =$ min. which can only happen when $y = 0$, $\therefore$ when $r =$ max. we must have $r = a - \frac{b}{n}\sqrt{m^2-n^2}$ or $a - r = \frac{b\sqrt{m^2-n^2}}{n}$ and $\therefore x$ $= \frac{A}{2} = \frac{mn(a-r)}{m^2-n^2} = \frac{mb}{\sqrt{m^2-n^2}}$, as before.

PROB. (10.) TO FIND THAT POINT (F) IN A GIVEN ELLIPSE *ABHD* WHICH, OF ALL OTHERS, IS THE MOST REMOTE FROM THE EXTREMITY B OF THE CONJUGATE AXIS. (Fig. 15.)

Drawing FE parallel to the transverse axis AH, and making $AH = a$, $BD = b$, and $BE = x$, we have, by the property of the curve $BF^2 = BE^2 + EF^2 = x^2 + \frac{a^2}{b^2}(bx - x^2)$ $= x^2 + \frac{a^2x}{b} - \frac{a^2}{b^2}x^2$. Now a is greater than b, $\therefore \frac{a^2}{b^2}$ must be greater than unity, and therefore $\left(1 - \frac{a^2}{b^2}\right)x^2 = \frac{b^2-a^2}{b^2}$ $x^2 = -\left(\frac{a^2-b^2}{b^2}\right)x^2$, $\therefore BF^2 = x^2 + \frac{a^2x}{b} - \frac{a^2}{b^2}x^2 = \frac{a^2}{b}x -$ $\left(\frac{a^2-b^2}{b^2}\right)x^2 = \left(\frac{a^2-b^2}{b^2}\right)\left(\frac{a^2b}{a^2-b^2}x - x^2\right) =$ max. and therefore $\frac{a^2b}{a^2-b^2}x - x^2 =$ max. which let $= r$, and we therefore find $x^2 - \frac{a^2b}{a^2-b^2}x = -r$. Solving this quadratic we find $x = \frac{\frac{1}{2}a^2b}{a^2-b^2} \pm \sqrt{\left(\frac{\frac{1}{2}a^2b}{a^2-b^2}\right)^2 - r}$. Now it is evident

that $r =$ max. when $\left(\frac{\frac{1}{2}a^2b}{a^2-b^2}\right)^2 = r$. But from the nature of the figure, the greatest value that x ($= BE$) can possibly admit of is $b = BD$, therefore if the relation of a and b be such that $\frac{\frac{1}{2}a^2b}{a^2-b^2}$ is greater than b, this solution is manifestly impossible. To determine this limit, therefore, make $\frac{\frac{1}{2}a^2b}{a^2-b^2} = b$; then it will be found that $2b^2 = a^2$. Whence the foregoing problem can only obtain when $2BD^2$ is equal to, or less than AH^2.

The same solved without impossible roots.

In the expression $\frac{a^2b}{a^2-b^2}x - x^2 =$ max. let $\frac{a^2b}{a^2-b^2} = A$ $\therefore Ax - x^2 =$ max. Let $x = y + \frac{A}{2}$, therefore $Ax - x^2$ $= Ay + \frac{A^2}{2} - y^2 - Ay - \frac{A^2}{4} = \frac{A^2}{4} - y^2 =$ max. when $y = 0$ $\therefore x = y + \frac{A}{2} = \frac{A}{2} = \frac{\frac{1}{2}a^2b}{a^2-b^2}$ as before.

PROB. (11.) GIVEN THE BASE AND PERPENDICULAR OF A TRIANGLE, TO DESCRIBE IT SO THAT THE VERTICAL ANGLE MAY BE A MAXIMUM. (Fig. 16.)

Let $AB = c$, $DC = p$, and $AD = x$, $\therefore DB = c - x$, $\frac{AD}{DC}$ $= \tan a = \frac{x}{p}$, $\frac{DB}{DC} = \tan b = \frac{c-x}{p}$ and $\therefore \tan C = \tan$

$$(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} = \frac{\frac{x}{p} + \frac{c-x}{p}}{1 - \frac{x(c-x)}{p^2}} = \frac{cp}{p^2 - cx + x^2}$$

$= \text{maximum} \therefore \frac{p^2 - cx + x^2}{cp} = \text{min.}$ which let $= r \therefore x^2 - cx = rpc - p^2$. Solving this quadratic we find $x = \frac{c}{2} + \sqrt{pc\left(r + \frac{c^2 - 4p^2}{4pc}\right)}$. This problem has three cases: 1st. Let $c \angle 2p$ and $\therefore \frac{c^2 - 4p^2}{4pc}$ must be a negative quantity, and therefore in this case r cannot be taken so small as to be less than this negative quantity, $\therefore$ when $r = \text{min.}$ it must $= \frac{4p^2 - c^2}{4pc}$ and $\therefore x = \frac{c}{2}$. 2nd. Let $c = 2p \therefore c^2 = 4p^2 \therefore \frac{c^2 - 4p^2}{4pc} = 0$, and $x = \frac{c}{2} + \sqrt{pcr}$. Now when r, or the co-tangent or the tangent of the complement of the vertical angle $C = \text{min.}$ it must $= 0$, or $x = \frac{c}{2}$. In this case, since the complement of $C = 0$, the angle itself must $= 90$ degrees. 3rd. Let $c \mathrel{7} 2p$ or $c^2 \mathrel{7} 4p^2$. In this case when r or the co-tangent of the angle $C = \text{min.}$ it must be a negative quantity, equal to the positive quantity $\frac{c^2 - 4p^2}{4pc}$, and $\therefore x = \frac{c}{2}$. In this third case it is evident that the vertical angle C must be obtuse, because its co-tangent is negative. It is also evident that in every case the triangle is isosceles.

The same solved without impossible roots.

In the expression $x^2 - cx + p^2 = \text{min.}$ let $x = y + \frac{c}{2} \therefore$

$x^2 - cx + p^2 = y^2 + cy + \frac{c^2}{4} - cy - \frac{c^2}{2} + p^2 = y^2 - \frac{c^2}{4} + p^2 = \text{min.}$ when $y = 0$, $\therefore x = \frac{c}{2}$ as before.

PROB. (12.) TO FIND THE POINT D IN THE STRAIGHT LINE CE, FROM WHICH AB SUBTENDS THE GREATEST ANGLE. (Fig. 17.)

Let $AC = a$, $CB = b$, and $CD = x$. It is evident that

$$\tan. ADB = \tan. (ADM - BDM) = \frac{\frac{AM}{MD} - \frac{BM}{MD}}{1 + \frac{MA\ MB}{MD^2}} =$$

$$\frac{(AM - BM)\ MD}{MD^2 + AM\ BM}.$$ It is also evident that $MD = x \sin \theta$, $AM = a - x \cos. \theta$ and $BM = b - x \cos. \theta$, we, therefore, find $\tan. \phi = \dfrac{(a-b)\, x \sin \theta}{x^2 \sin^2 \theta + (a - x \cos. \theta)\ (b - x \cos. \theta)}$ a maximum $\therefore y = \dfrac{x^2 \sin^2 \theta + (a - x \cos. \theta)\ (b - x \cos. \theta)}{(a-b)\, x \sin \theta}$ a minimum; and since $(a - b) \sin \theta$ is a constant given quantity $\dfrac{x^2 \sin^2 \theta + (a - x \cos \theta)\ (b - x \cos \theta)}{x}$ must also be a minimum which let $= r$, $\therefore x^2 \sin^2 \theta + ab - (a+b) \cos \theta\, x + \cos^2 \theta \cdot x^2 = x^2 (\sin^2 \theta + \cos^2 \theta) + ab - (a+b) \cos \theta \cdot x = x^2 + ab - (a+b) \cos \theta \cdot x = rx$, therefore $x^2 - \left\{(a+b) \cos \theta + r\right\} x = -ab$. Solving this quadratic we find

$$x = \frac{(a+b) \cos \theta + r}{2} \pm \sqrt{\left\{\frac{(a+b) \cos \theta + r}{2}\right\}^2 - ab}.$$

Now r cannot be taken so small (or, if necessary, negatively so great) as to make $\left\{\dfrac{(a+b) \cos \theta + r}{2}\right\}^2$ less than ab, because this supposition makes the value of x impossible $\therefore$ when $r = \text{min.}$ we must have $\left\{\dfrac{(a+b) \cos \theta + r}{2}\right\}^2 = ab$, $\therefore r = 2\sqrt{ab} - (a+b) \cos \theta$ and $x = \dfrac{(a+b) \cos \theta + r}{2}$ $= \dfrac{2\sqrt{ab}}{2} = \sqrt{ab}$.

The same solved without impossible roots.

In the expression $x^2 - \left\{(a + b)\cos\theta + r\right\}x = -ab$ let the co-efficient of $x = A$ and let $x = y + \frac{A}{2}$, we therefore find $x^2 - Ax + ab = y^2 + Ay + \frac{A^2}{4} - Ay - \frac{A^2}{2} + ab = y^2 - \frac{A^2}{4} + ab = y^2 + ab - \frac{A^2}{4} = 0$, or $\frac{A^2}{4} = y^2 + ab$. Now it is evident that r and $\frac{A^2}{4} = \text{min.}$ when $y = 0$, $\therefore$ $\frac{A}{2} = \sqrt{ab}$ $\therefore x = \sqrt{ab}$ as before.

PROB. (13.) TO BISECT A TRIANGLE BY THE SHORTEST LINE. (Fig. 18.)

Let ABC be the given triangle, and PQ the shortest line required. Also let $CP = x$, $CQ = y$, $PQ = u$, and a, b, c the three sides of the triangle, and C the angle BCA. Pm and Bn are perpendiculars drawn from the points P and B on the line CA. Now by similar triangles we find $\frac{Pm}{CP} = \frac{Bn}{CB} = \sin C$, $\therefore$ $Pm = x \sin C$ and $Bn = a \sin C$ and $\therefore$ $\frac{CQ \times Pm}{2} = \frac{xy \sin C}{2}$ and $\frac{CA \times Bn}{2} = \frac{ab \sin C}{2}$; but by supposition $2 \times \frac{CQ \times Pm}{2} = \frac{CA \times Bn}{2}$ $\therefore$ $2 \times \frac{xy \sin C}{2} = \frac{ab \sin C}{2}$ $\therefore$ $ab = 2xy$ $\therefore$ $y = \frac{ab}{2x}$.

By Prop. 18, Book 2d of Euclid we find—

$u^2 = x^2 + y^2 - 2xy \cos C = x^2 + \frac{a^2b^2}{4x^2} - ab \cos C = \text{min.}$

which let $= r$, $\therefore x^4 + \frac{a^2b^2}{4} - ab \cos C . x^2 = rx^2$, and therefore $x^4 - (ab \cos C + r)\ x^2 = -\frac{a^2b^2}{4}$. Completing the square and extracting the square root we find,
$x^2 = \frac{ab \cos C + r}{2} \pm \sqrt{\frac{(ab \cos C + r)^2 - a^2b^2}{4}}$. Now a^2b^2 is greater than $a^2b^2 \cos^2 C$, $\therefore$ in order that the value of x^2 may not become impossible, we must have $ab \cos C + r = ab$, $\therefore r = ab - ab \cos C$, and $\therefore$ when $r =$ min. we must have $x^2 = \frac{ab \cos C + r}{2} = \frac{ab}{2} \therefore x = \sqrt{\frac{ab}{2}}$ and $y = \frac{ab}{2x} = \sqrt{\frac{ab}{2}} \therefore u^2 = \frac{ab}{2} + \frac{ab}{2} - ab \cos C = ab\ (1 - \cos C) = ab \left\{\frac{2ab + c^2 - (a^2 + b^2)}{2ab}\right\} = \frac{c^2 - (a - b)^2}{2}$ and $\therefore u = \sqrt{\frac{(c - a + b)\ (c + a - b)}{2}}$.

The same solved without impossible roots.

Let $ab \cos C + r = A$ and $\frac{a^2b^2}{4} = B \therefore$ the equation $x^4 - (ab \cos C + r)\ x^2 = -\frac{a^2b^2}{4}$ becomes $x^4 - Ax^2 = -B$. Also let $x^2 = y + \frac{A}{2}$, $\therefore x^4 - Ax^2 = y^2 + Ay + \frac{A^2}{4} - Ay - \frac{A^2}{2} = y^2 - \frac{A^2}{4} = -B$, $\therefore y^2 + B = \frac{A^2}{4} \therefore$ when $\frac{A^2}{4}$ and $r =$ min. $y = 0$, $\therefore B = \frac{A^2}{4}$ or $\frac{a^2b^2}{4} = \frac{A^2}{4}$ and $ab = A = ab \cos C + r$ and $r = ab - ab \cos C = ab\ (1 - \cos C)$ and $x^2 = \frac{A}{2} = \frac{ab \cos C + r}{2} = \frac{ab}{2}$, $\therefore x = \sqrt{\frac{ab}{2}}$ as before.

PROB. (14.)

Let $y = x \tan\theta - \frac{x^2}{p \cos^2\theta}$; find x that y may be a maximum. Now $y = \frac{4p \cos^2\theta \tan\theta \cdot x - x^2}{4p \cos^2\theta} =$ max. and since $4p \cos^2\theta$ is a constant given quantity, we must have $4p \cos^2\theta \tan\theta\, x - x^2 =$ max. which let $= r$. Also let the coefficient of x in this equation $= 2A$, and we therefore find $2Ax - x^2 =$ max. $= r$ or $2Ax - x^2 = r$ and hence $x^2 - 2Ax = -r$. Solving this quadratic we find $x = A + \sqrt{A^2 - r}$, $\therefore$ when $r =$ max. we must have $A^2 = r$ $\therefore$ $x = A = \frac{2A}{2} = \frac{4p \cos^2\theta \tan\theta}{2} = 2p \cos^2\theta \tan\theta = 2p \sin\theta \cos\theta$ and we find $y = 2p \tan\theta \sin\theta \cos\theta - \frac{4p^2 \sin^2\theta \cos^2\theta}{4p \cos^2\theta} = 2p \sin^2\theta - p \sin^2\theta = p \sin^2\theta$. The equation is that of the path of a projectile, and the maximum value of y is the greatest altitude above the horizontal plane.

The same solved without impossible roots.

In the expression $2Ax - x^2 =$ max. let $x = y + A$, $\therefore$ $2Ax - x^2 = 2Ay + 2A^2 - y^2 - 2Ay - A^2 = A^2 - y^2$ which is evidently $=$ max. when $y = 0$ $\therefore$ $x = A = \frac{2A}{2} = 2p \sin\theta \cos\theta$ as before.

PROB (15.) DIVIDE A NUMBER a INTO TWO SUCH FACTORS THAT THE SUM OF THEIR SQUARES SHALL BE A MINIMUM.

Let $x =$ one of the factors, $\therefore \frac{a}{x} =$ the other factor, and

their squares $= x^2 + \frac{a^2}{x^2} =$ min. $= r$ $\therefore$ $x^4 + a^2 = rx^2$, and $\therefore$ $x^4 - rx^2 = - a^2$. Solving this quadratic we find, $x^2 = \frac{r}{2} \pm \sqrt{\frac{r^2}{4} - a^2}$. It is now evident that when $r =$ min. we must have $\frac{r^2}{4} = a^2$ or $\frac{r}{2} = a$, $\therefore$ $x^2 = \frac{r}{2} = a$ $\therefore$ $x = \sqrt{a}$.

The same solved without impossible roots.

In the expression $x^4 - rx^2 = - a^2$ suppose $x^2 = y + \frac{r}{2}$ $\therefore$ $x^4 - rx^2 = y^2 + ry + \frac{r^2}{4} - ry - \frac{r^2}{2} = v^2 - \frac{r^2}{4} = - a^2$ $\therefore$ $y^2 + a^2 = \frac{r^2}{4}$, which is evidently a minimum when $y = 0$, $\therefore$ $\frac{r}{2} = a$ and $x^2 = \frac{r}{2} = a$, $\therefore$ $x = \sqrt{a}$ as before.

PROB. (16.) FIND THAT FRACTION WHICH EXCEEDS ITS SECOND POWER BY THE GREATEST POSSIBLE NUMBER.

Let x be the fraction, and it is required to find such a value for x which may make $x^1 - x^2$ a maximum. Let $x - x^2 = r$ $\therefore$ $x^2 - x = - r$, and solving this quadratic we find $x = \frac{1}{2} + \sqrt{\frac{1}{4} - r}$. Now it is evident that r cannot be greater than $\frac{1}{4}$, and therefore when $r =$ max. it must be $= \frac{1}{4}$ and $\therefore$ $x = \frac{1}{2} =$ the fraction required.

The same solved without impossible roots.

In the expression $x - x^2 =$ maximum, let $x = y +$ the coefficient of $\frac{x^2}{2} = y + \frac{1}{2}$, and $\therefore$ we find $x - x^2 = y + \frac{1}{2} - y^2 - y - \frac{1}{4} = \frac{1}{4} - y^2$, which is evidently a maximum when $y = 0$, $\therefore$ $x = \frac{1}{2}$ as before.

PROB. (17.) OF ALL TRIANGLES UPON THE SAME BASE, AND HAVING THE SAME PERIMETER, FIND THAT WHICH HAS THE GREATEST AREA.

Let $2P$ be the perimeter, a the given base, x and y the remaining sides. It is demonstrated in the Introduction that in any plane triangle whose sides are a, x and y and semi-perimeter $= P$, the area $= \sqrt{P(P-a)(P-x)(P-y)}$; and because the square of a maximum is a maximum, we must have $P(P-a)(P-x)(P-y) = \max.$ and $P(P-a)$ is a given constant quantity, we must also have $(P-x)(P-y) = \max.$ Now $y = 2P - a - x \therefore P - y = P - 2P + a + x = a + x - P$, therefore by substitution we find $(P-x)(a+x-P) = \max.$ and $\therefore aP - P^2 + (2P-a)x - x^2 = \max.$ and as $aP - P^2$ is a constant given quantity, we must also have $(2P-a)x - x^2 = \max.$ which let $= r$. We now have $x^2 - (2P-a)x = -r$, and solving this quadratic we find $x = \frac{2P-a}{2} + \sqrt{\frac{(2P-a)^2}{4} - r}$. It is evident that when r is a maximum, it must be $= \frac{(2P-a)^2}{4} \therefore x = \frac{2P-a}{2} = P - \frac{a}{2}$ and $y = 2P - a - x = 2P - a - P + \frac{a}{2} = P - \frac{a}{2}$ and therefore $y = x$, or the triangle is isosceles.

The same solved without impossible roots.

In the expression $(2P-a)x - x^2 = \max.$ let $x = \frac{2P-a}{2} + y$, $\therefore (2P-a)x - x^2 = (2P-a)y + \frac{(2P-a)^2}{2} - y^2 - (2P-a)y - \frac{(2P-a)^4}{4} = \frac{(2P-a)^2}{4} - y^2 =$

max. which happens when $y = 0 \therefore x = \frac{2P - a}{2} = P - \frac{a}{2}$ as before.

PROB. (18.) TO INSCRIBE THE GREATEST PARALLELOGRAM WITHIN A GIVEN TRIANGLE ABC, THE ANGLE A BEING ONE OF THE ANGLES OF THE PARALLELOGRAM. (Fig. 19.)

Let $AEGF$ be the greatest inscribed parallelogram required, and ED the perpendicular let fall from one of its angles E, upon one of its sides AF. Also let $AB = c$, $AC = b$ and $AE = x$.

The area of the parallelogram $= AF \times ED$. The lines EG and AC being parallel, the triangles ABC and EBG must be similar, and consequently $AB : EB :: AC : EG$, or $c : AB - AE :: b : AF$; or $c : c - x :: b : AF, \therefore AF = \frac{b\,(c - x)}{c}$ and the perpendicular ED is evidently $= EA \times \sin A = x \sin A$. Now substituting these values of AF and ED, we find area of the parallelogram $= \frac{x \sin A \times b\,(c - x)}{c} = \frac{b \sin A}{c}(cx - x^2) = \text{max.}$; and since $\frac{b \sin A}{c}$ is a constant given quantity, we must also have $cx - x^2 = \text{max.}$ Let $cx - x^2 = \text{max.} = r \therefore x^2 - cx = -r$, and therefore $x = \frac{c}{2} \pm \sqrt{\frac{c^2}{4} - r}$, and hence it is evident that when $r = \text{max.}$ it must be $= \frac{c^2}{4}$, and $\therefore x = \frac{c}{2}$ or $AE = \frac{AB}{2}$.

The same solved without impossible roots.

In the expression $cx - x^2 = \text{max.}$ let $x = y + \frac{c}{2}$ and

$\therefore\ cx - x^2 = cy + \frac{c^2}{2} - y^2 - cy - \frac{c^2}{4} = \frac{c^2}{4} - y^2$ which is evidently $=$ max. when $y = 0$, $\therefore\ x = \frac{c}{2}$ as before.

PROB. (19.) OF ALL EQUI-ANGULAR AND ISOPERIMETRICAL PARALLELOGRAMS FIND THAT WHICH HAS THE GREATEST AREA. (Fig. 20.)

Let $ACDE$ be the required parallelogram, $AE = x$, $AC = y$, and semi-perimeter $= a$. It is evident that the area of this parallelogram $= AC \times EB$ .... (1.)

Now by supposition $x + y = a$, $\therefore\ y = a - x = AC$ and $EB = AE \sin A = x \sin A$; substituting these values of AC and EB in equation (1) we find, area of the parallelogram $= \sin A\ (ax - x^2) =$ max. Now as $\sin A$ is a constant given quantity, we must have also $ax - x^2 =$ max. which let $= r$ $\therefore\ x^2 - ax = -r$. Solving this quadratic we find $x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - r}$, and it is evident from this value of x, that when $r =$ max. we must have $r = \frac{a^2}{4}$ $\therefore\ x = \frac{a}{2}$ and $y = a - x = a - \frac{a}{2} = \frac{a}{2}$ $\therefore\ x = y$. Hence it appears that of all equi-angular and isoperimetrical parallelograms, the equi-lateral has the greatest area.

The same solved without impossible roots.

In the expression $ax - x^2 =$ max. let $x = y + \frac{a}{2}$ and therefore $ax - x^2 = ay + \frac{a^2}{2} - y^2 - ay - \frac{a^2}{4} = \frac{a^2}{4} - y^2 =$ max. when $y = 0$, $\therefore\ x = \frac{a}{2}$ as before.

PROB. (20.) OF ALL TRIANGLES ON THE SAME BASE, AND HAVING EQUAL VERTICAL ANGLES, TO FIND THAT WHICH HAS THE GREATEST PERIMETER. (Fig. 21.)

Let ABC be the required triangle, of which the base AC is given $= b$, and the vertical angle $ABC = B$: it is required to find the mutual relation and magnitudes of the remaining sides $AB = x$ and $BC = y$ when the perimeter or the sum of all the sides is a maximum. Let a perpendicular AD be drawn to the line BC. It is evident, by the first principles of trigonometry, that $BD = AB \cos B = x \cos B$, $AD = AB \sin B = x \sin B$ and $\therefore\ DC = \sqrt{AC^2 - AD^2} = \sqrt{b^2 - x^2 \sin^2 B}$, and $\therefore\ y = BD + DC = x \cos B + \sqrt{b^2 - x^2 \sin^2 B}$ $\therefore$ perimeter $= b + x + x \cos B + \sqrt{b^2 - x^2 \sin^2 B} = b + (1 + \cos B)\, x + \sqrt{b^2 - x^2 \sin^2 B}$ $=$ max. and as b is a constant given quantity, we must also have $(1 + \cos B)\, x + \sqrt{b^2 - x^2 \sin^2 B} =$ max. which let $= r$ $\therefore\ \sqrt{b^2 - x^2 \sin^2 B} = r - (1 + \cos B)\, x$, and, squaring both sides, we find $b^2 - x^2 \sin^2 B = r^2 - 2r\,(1 + \cos B)\, x + (1 + \cos B)^2 x^2$, therefore $\left\{\sin^2 B + (1 + \cos B)^2\right\} x^2 - 2\,(1 + \cos B)\, rx = b^2 - r^2$ $\therefore\ x^2 - \dfrac{2\,(1 + \cos B)\, r}{\sin^2 B + (1 + \cos B)^2}\, x = \dfrac{b^2 - r^2}{\sin^2 B + (1 + \cos B)^2}$. Solving this quadratic we find $x =$

$$\frac{(1 + \cos B)\, r}{\sin^2 B + (1 + \cos B)^2} \pm \sqrt{\frac{\left\{\sin^2 B + (1 + \cos B)^2\right\} b^2 - \sin^2 B\, r^2}{\left\{\sin^2 B + (1 + \cos B)^2\right\}^2}}$$

Now it is evident that r or $\sin^2 B\, r^2$ when a maximum, must be $= \left\{\sin^2 B + (1 + \cos B)^2\right\} b^2$ or $r = \dfrac{b\sqrt{\sin^2 B + (1 + \cos B)^2}}{\sin B}$

and therefore we find $x = \dfrac{b\,(1 + \cos B)}{\sin B\sqrt{\sin^2 B + (1 + \cos B)^2}}$

and $y = x\cos B + \sqrt{b^2 - x^2\sin^2 B} = \dfrac{b\,(1+\cos B)\cos B}{\sin B\sqrt{\sin^2 B + (1+\cos B)^2}}$
$r\,\dfrac{b\sin B}{\sqrt{\sin^2 B + (1+\cos B)^2}} = \dfrac{b\cos B + b\cos^2 B + b\sin^2 B}{\sin B\sqrt{\sin^2 B + (1+\cos B)^2}}$
$= \dfrac{b\,(1+\cos B)}{\sin B\sqrt{\sin^2 B + (1+\cos B)^2}} \therefore x = y.$ Hence of all triangles on the same base, having equal vertical angles, the isosceles has the greatest perimeter.

The same solved without impossible roots.

In the equation $x^2 - \dfrac{2\,(1+\cos B)\,r}{\sin^2 B + (1+\cos B)^2}\,x = \dfrac{b^2}{\sin^2 B + (1+\cos B)^2} - \dfrac{r^2}{\sin^2 B + (1+\cos B)^2}$, let $\dfrac{(1+\cos B)}{\sin^2 B + (1+\cos B)^2} = m$, $\dfrac{b^2}{\sin^2 B + (1+\cos B)^2} = n$ and $\dfrac{1}{\sin^2 B + (1+\cos B)^2} = q \therefore x^2 - 2mr\,x = n - qr^2$. Also let $x = y + mr$ and we therefore find $y^2 + 2mry + m^2r^2 - 2mry - 2m^2r^2 = y^2 - m^2r^2 = n - qr^2 \therefore r^2 = \dfrac{n - y^2}{q - m^2}$ which is evidently $=$ max. when $y = 0$, $\therefore r = \dfrac{\sqrt{n}}{\sqrt{q - m^2}}$ and $\therefore x = \dfrac{m\sqrt{n}}{\sqrt{q - m^2}} = \dfrac{b\,(1+\cos B)}{\sin B\sqrt{\sin^2 B + (1+\cos B)^2}}$ by substitution as before.

PROB. (21.) TO INSCRIBE THE GREATEST RECTANGLE IN A GIVEN SEMICIRCLE. (Fig. 22.)

Let $CN = x$, and $CA = a \therefore NP = \sqrt{a^2 - x^2}$ and therefore the rectangle required $= 2PM \times CM = 2x\sqrt{a^2 - x^2} =$ max. $\therefore 4a^2x^2 - 4x^4 =$ max. and $\therefore a^2x^2 - x^4 =$ max. which

let $= r$, $\therefore x^4 - a^2x^2 = -r$, $\therefore x^2 = \frac{a^2}{2} \pm \sqrt{\frac{a^4}{4} - r^2}$. It is evident that when $r =$ max. it is $= \frac{a^4}{4} \therefore x^2 = \frac{a^2}{2} \therefore x = \frac{a}{\sqrt{2}}$.

The same solved without impossible roots.

In the expression $a^2x^2 - x^4 =$ max. let $x^2 = y + \frac{a^2}{2} \therefore a^2x^2 - x^4 = a^2y + \frac{a^4}{2} - y^2 - a^2y - \frac{a^4}{4} = \frac{a^4}{4} - y^2 =$ max. when $y^2 = 0$, $\therefore x^2 = \frac{a^2}{2}$ and $x = \frac{a}{\sqrt{2}}$ as before.

PROB. (22.) OF ALL SQUARES INSCRIBED IN A GIVEN SQUARE TO FIND THAT WHICH IS THE LEAST. (Fig. 23.)

Let $ABCD$ be the given square, and $abcd$ the required one. Also let $AB = BC = a$, $aB = x$, $\therefore Aa = a - x$. Now it is evident that $ab = ac$, the $\angle A = \angle B$ and the angles Aab and Aba are together equal to 90 degrees $=$ angles Aab and Bac $\therefore \angle Aba = \angle Bac$ $\therefore$ the third angle $Aab = \angle Bca$ $\therefore Aa = Bc$; but $Aa = a - x$ $\therefore Bc = a - x$. Now it is evident that $aB^2 + Bc^2 = ac^2$ or $x^2 + (a - x)^2 = ac^2$ $=$ the area of the square required $=$ a maximum, which let $= r$, $\therefore 2x^2 - 2ax + a^2 = r$, and by proceeding exactly as in problem (5) we find $x = \frac{a}{2}$ when $r =$ max.

The same may be solved without impossible roots as in problem (5.)

PROB. (23.) TO INSCRIBE THE GREATEST RECTANGLE IN A GIVEN ELLIPSE. (Fig. 24.)

Let $AFGBED$ be the given Ellipse, and $FDEG$ the inscribed rectangle required. Also let mC (where C is the centre) $= Cn = x$, $AC = a$ and $pC = b$ $\therefore$ $mn = 2x$. Now by the property of the Ellipse demonstrated in the Introduction we find $mF = \frac{b}{a}\sqrt{a^2 - x^2}$ $\therefore$ $2mF = \frac{2b}{a}\sqrt{a^2 - x^2}$ and therefore the rectangle $FE = FD \times DE = FD \times mn = \frac{2b}{a}\sqrt{a^2 - x^2} \times 2x = \frac{4b}{a}\sqrt{a^2x^2 - x^4} =$ max., and as $\frac{4b}{a}$ is a constant given quantity, we find $\sqrt{a^2x^2 - x^4} =$ max. and also $a^2x^2 - x^4 =$ max. which let $= r$, $\therefore$ $x^4 - a^2x^2 = -r$. Solving this quadratic we find $x^2 = \frac{a^2}{2} \pm \sqrt{\frac{a^4}{4} - r}$, and hence it is manifest that r cannot be greater than $\frac{a^4}{4}$ and therefore when $r =$ max. it must be $= \frac{a^4}{4}$ and $\therefore$ $x^2 = \frac{a^2}{2}$ and $x = \frac{a}{\sqrt{2}}$.

This problem may be solved without impossible roots, exactly in the same way as problem (21.)

PROB. (24.) GIVEN THE BASE AND THE VERTICAL ANGLE OF A TRIANGLE, SHOW THAT WHEN IT IS ISOSCELES ITS AREA IS A MAXIMUM. (See Fig. 21.)

Let ABC be the required triangle of which the base AC is given $= b$, and the vertical angle $ABC = B$: it is required to find the mutual relation of the remaining sides $AB = x$

and $BC = y$ when the area of the triangle is a maximum. Let a perpendicular AD be drawn to the line BC. It is evident, by the first principles of Trigonometry, that $BD = AB \cos B = x \cos B$, $AD = AB \sin B = x \sin B$ and $\therefore$ $DC = \sqrt{AC^2 - AD^2} = \sqrt{b^2 - x^2 \sin^2 B}$ and $\therefore$ $y = BD + DC = x \cos B + \sqrt{b^2 - x^2 \sin^2 B}$, therefore the area of the triangle $ABC = AD \times BC = x \sin B \left\{x \cos B + \sqrt{b^2 - x^2 \sin^2 B}\right\}$ $= \sin B\ (x^2 \cos B + x\sqrt{b^2 - x^2 \sin^2 B}) =$ max. Now as $\sin B$ is a constant given quantity, we must have $x^2 \cos B + x\sqrt{b^2 - x^2 \sin^2 B} =$ max. which let $= r$, $\therefore$ $x\sqrt{b^2 - x^2 \sin^2 B}$ $= r - x^2 \cos B$ or $b^2 x^2 - x^4 \sin^2 B = r^2 - 2r \cos B\, x^2 + x^4 \cos^2 B$ or $x^4 (\cos^2 B + \sin^2 B) - (b^2 + 2r \cos B)\, x^2 = - r^2$ or $x^4 - (b^2 + 2r \cos B)\, x^2 = - r^2$. Solving this quadratic we find $x^2 = \dfrac{b^2 + 2r \cos B}{2} \pm \sqrt{\dfrac{(b^2 + 2r \cos B)^2}{4} - r^2}$ $= \dfrac{b^2 + 2r \cos B}{2} \pm \sqrt{\dfrac{b^4 - 4r\,(r \sin^2 B - b^2 \cos B)}{4}}$.

Now it is evident that r cannot be taken so great as to make $4r\,(r \sin^2 B - b^2 \cos B)$ greater than b^4, and therefore when $r =$ max. we must have $b^4 = 4r\,(r \sin^2 B - b^2 \cos B)$ and from this equation we find $r^2 - \dfrac{b^2 \cos B}{\sin^2 B}\, r = \dfrac{b^4}{4 \sin^2 B}$ and, solving this quadratic, we find $2r = \dfrac{b^2\,(1 + \cos B)}{\sin^2 B}$. Substituting this value of $2r$ in the equation $x^2 = \dfrac{b^2 + 2r \cos B}{2}$ we find $x^2 = \dfrac{b^2\,(1 + \cos B)}{2 \sin^2 B}$ and $x = \sqrt{\dfrac{b^2\,(1 + \cos B)}{2 \sin^2 B}}$ and $y = BD + DC = x \cos B + \sqrt{b^2 - x^2 \sin^2 B} =$ $\cos B \sqrt{\dfrac{b^2\,(1 + \cos B)}{2 \sin^2 B}} + \sqrt{b^2 - \dfrac{b^2\,(1 + \cos B) \sin^2 B}{2 \sin^2 B}} =$ $\cos B \sqrt{\dfrac{b^2\,(1 + \cos B)}{2 \sin^2 B}} + \sqrt{\dfrac{b^2\,(1 - \cos B)}{2}}$, $\therefore$ $y =$

$\cos B \sqrt{\dfrac{b^2(1+\cos B)}{2\sin^2 B}} + \sqrt{\dfrac{b^2(1-\cos B)}{2}}$; squaring both sides of this equation we find $y^2 = \dfrac{\cos^2 B}{\sin^2 B} \cdot \dfrac{b^2(1+\cos B)}{2}$

$$+ 2\cos B \sqrt{\frac{b^4 \sin^2 B}{4\sin^2 B}} + \frac{b^2 \ (1 - \cos B)}{2} = b^2 \cos B$$

$$+ b^2 \frac{\left\{\cos^2 B\ (1+\cos B) + \sin^2 B\ (1-\cos B)\right\}}{2\sin^2 B} = b^2 \cos B$$

$$+ \frac{b^2\ (1 + \cos^3 B - \sin^2 B \cos B)}{2 \sin^2 B} = b^2 \cos B +$$

$$\frac{\left\{1 + \cos B\ (1 - \sin^2 B) - \cos B \sin^2 B\right\}}{2\sin^2 B} b^2$$

$$= b^2 \left\{\cos B + \frac{1 + \cos B - 2\sin^2 B \cos B}{2 \sin^2 B}\right\}$$

$= \dfrac{b^2(1+\cos B)}{2\sin^2 B} = x^2$, $\therefore y^2 = x^2$, and $\therefore y = x$. Hence it appears that the triangle must be isosceles, in order that its area may be a maximum.

PROB. (25.) TO FIND THE LEAST TRIANGLE TCt, WHICH CAN BE DESCRIBED ABOUT A GIVEN QUADRANT. (Fig. 25.)

Let $CA = a$, $tC = x$, and $CT = y$. It is evident that the line or hypothenuse Tt is a tangent to the quadrant at the point P, and therefore the angles tPC and CPT are right angles. By similar triangles, according to Prop. 8, Book 6 of Euclid, we have $tC : CP :: CP : CN$ or $x : a :: a : CN$ $\therefore$ $CN = \dfrac{a^2}{x}$ and $NP = CM = \sqrt{CB^2 - CN^2} = \sqrt{a^2 - \dfrac{a^4}{x^2}} = \dfrac{a}{x}\sqrt{x^2 - a^2}$. Also $CT : CP :: CP : CM$ or $y : a :: a : \dfrac{a}{x}\sqrt{x^2 - a^2}$, $\therefore y = \dfrac{ax}{\sqrt{x^2 - a^2}}$ and therefore

the area of the triangle $TCt = \frac{1}{2}xy = \frac{1}{2}a \times \frac{x^2}{\sqrt{x^2 - a^2}} =$ minimum. Now $\frac{1}{2}a$ is a constant given quantity, $\therefore \frac{x^2}{\sqrt{x^2-a^2}}$ or its square $\frac{x^4}{x^2 - a^2} =$ min. which let $= r$, $\therefore x^4 = rx^2 - ra^2$ or $x^4 - rx^2 = - ra^2$..................................(1)

Solving this quadratic we find $x = \frac{r}{2} \pm \sqrt{\frac{r^2 - 4ra^2}{4}} = \frac{r}{2} \pm \sqrt{\frac{r\,(r - 4a^2)}{4}}$ and here it is evident that r cannot be less than $4a^2$, $\therefore$ it must be $= 4a^2$ when it is a minimum, $\therefore x^2 = \frac{r}{2} = \frac{4a^2}{2} = 2a^2$ and $x = a\sqrt{2}$ and $y = \frac{ax}{\sqrt{x^2 - a^2}} = \frac{a^2\sqrt{2}}{a} = a\sqrt{2}$, $\therefore x = y$. Hence it appears that the angle PTC must be $=$ 45 degrees, or that the triangle described must be isosceles when it is the least possible.

The same solved without impossible roots.

In the equation (1) viz. in $x^4 - rx^2 = - ra^2$ let $x^2 = y + \frac{r}{2}$ and therefore $x^4 - rx^2 = y^2 + ry + \frac{r^2}{4} - ry - \frac{r^2}{2} = y^2 - \frac{r^2}{4} = - ra^2$, $\therefore r^2 - 4ra^2 = 4y^2$, and therefore we find $r = 2a^2 + \sqrt{4y^2 + 4a^4}$, and here it is evident that when $r =$ min. we must have $4y^2$ or $y = 0$, $\therefore r = 2a^2 + 2a^2 = 4a^2$ and $x^2 = \frac{r}{2} = \frac{4a^2}{2} = 2a^2$, $\therefore x = a\sqrt{2}$ as before.

PROB. (26.) SUPPOSING A SHIP TO SAIL FROM A GIVEN PLACE *A*, IN A GIVEN DIRECTION *AQ*, AT THE SAME TIME THAT A BOAT FROM ANOTHER GIVEN PLACE *B*, SETS OUT IN ORDER (IF POSSIBLE) TO COME UP WITH HER, AND SUPPOSING THE RATE AT WHICH EACH VESSEL PROGRESSES TO BE GIVEN, IT IS REQUIRED TO FIND IN WHAT DIRECTION, THE LATTER MUST PROCEED, SO THAT IF IT CANNOT COME UP WITH THE FORMER, IT MAY HOWEVER APPROACH IT AS NEAR AS POSSIBLE. (Fig. 26.)

Let the celerity of the ship be to that of the boat in the given ratio of m to n; also let D and F be the places of the two vessels when nearest possible to each other, and, from the centre B, through F, suppose the circumference of a circle to be described. Then the distance DF, being the least possible, the point F must be in the right line DB, joining the point D and the centre B; because no other point in the whole periphery, at which the boat from B might arrive in the same time, is so near to D as that wherein the line DB intersects the said periphery. But now, to get an expression for DF, in algebraic terms, let BC be perpendicular to AQ and make $AC = a$, $BC = b$, $CD = x$, and then BD will be $= \sqrt{BC^2 + CD^2} = \sqrt{b^2 + x^2}$; moreover, because $m : n :: AD$ or $a+x : BF$, we will have $BF = \frac{na+nx}{m}$, and consequently $DF = \sqrt{b^2 + x^2} - \frac{na + nx}{m} = \sqrt{b^2 + x^2} - \frac{na}{m} - \frac{nx}{m} =$ min. which let $= q$, $\therefore \sqrt{b^2 + x^2} - \frac{nx}{m} = q + \frac{na}{m}$ which let $= r$. Now it is evident that since $\frac{na}{m}$ is a constant given quantity, and $q =$ min. we must also have $q + \frac{na}{m}$ or $r =$ min. $\therefore \sqrt{b^2 + x^2} - \frac{nx}{m} =$ min. $= r$ or $\sqrt{b^2 + x^2}$

$= r + \frac{nx}{m}$ and therefore $b^2 + x^2 = r^2 + \frac{2nr}{m}x + \frac{n^2x^2}{m^2}$

$\frac{m^2-n^2}{m^2}x^2 - \frac{2nrx}{m} = r^2 - b^2$ or $x^2 - \frac{2rmn}{m^2-n^2}x = \frac{(r^2-b^2)m^2}{m^2-n^2}$ ......(1)

Solving this quadratic we find,

$$x = \frac{mnr}{m^2 - n^2} \pm \sqrt{\frac{(r^2 - b^2)\ m^2\ (m^2 - n^2) + m^2n^2r^2}{(m^2 - n^2)^2}} =$$

$$\frac{mnr}{m^2 - n^2} \pm \sqrt{\frac{\{\ m^2\ (m^2 - n^2) + m^2n^2\ \}\ r^2 - b^2m^2\ (m^2 - n^2)}{(m^2 - n^2)^2}}$$

$\frac{mnr}{m^2 - n^2} \pm \sqrt{\frac{m^4r^2 - b^2m^2\ (m^2 - n^2)}{(m^2 - n^2)^2}}$. Here it must be remarked, that this problem becomes impossible when m is less than n, for in this case the quantity $- b^2(m^2 - n^2)m^2$ must become a positive quantity, and therefore there remains no condition of r becoming a minimum. Now it is evident that m^4r^2 or r cannot be taken so small as to make the root impossible, therefore when $r = \text{min.}$ we must have $m^4r^2 = b^2m^2(m^2 - n^2)$ and $\therefore\ r = \frac{b\sqrt{m^2 - n^2}}{m}$ and $x = \frac{mnr}{m^2 - n^2} = \frac{nb}{\sqrt{m^2 - n^2}}$; also $DF = \sqrt{b^2 + x^2} - \frac{na + nx}{m} = r - \frac{na}{m} = \frac{b\sqrt{m^2 - n^2}}{m} - \frac{na}{m} = \frac{b\sqrt{m^2 - n^2} - na}{m}$; whence the position of F is known. From the above it is observable that, as DF must be a real positive quantity (by the question), this method of solution can only be of use when m is greater than n, and $b\sqrt{m^2-n^2}$, also greater than na: for in all other cases the boat will be able to come up with the ship.

The same solved without impossible roots.

In the equation (1) or $x^2 - \frac{2mnr}{m^2 - n^2}x = \frac{(r^2 - b^2)m^2}{m^2 - n^2}$ let half the co-efficient of $x = A$, and the second member of

the equation $= B$, $\therefore x^2 - 2Ax = B$. Now let $x = A + y$ $\therefore x^2 - 2Ax = y^2 + 2Ay + A^2 - 2Ay - 2A^2 = y^2 - A^2$ $= B$, $\therefore y^2 = B + A^2$, and by substitution, $y^2 = B + A^2 =$ $\frac{(r^2 - b^2)m^2}{m^2 - n^2} + \frac{m^2n^2r^2}{(m^2 - n^2)^2} = \frac{m^2n^2r^2 + (r^2 - b^2)m^2(m^2 - n^2)}{(m^2 - n^2)^2}$ $= \frac{m^4r^2 - b^2m^2(m^2 - n^2)}{(m^2 - n^2)^2}$ $\therefore m^4r^2 - b^2m^2(m^2 - n^2) =$ $(m^2 - n^2)^2y^2$, and therefore we find $r^2 = \frac{(m^2 - n^2)^2y^2 + b^2m^2(m^2 - n^2)}{m^4}$ which is evidently a minimum when y^2 or $y = 0$, $\therefore r =$ $\frac{b\sqrt{m^2 - n^2}}{m}$ and $x = \frac{nb}{\sqrt{m^2 - n^2}}$ as before.

PROB. (27.) TO FIND SUCH A VALUE FOR x AS WILL MAKE $b - (x - a)^2$ A MAXIMUM.

Let $b^2 - (x - a)^2 = r$, $\therefore b - x^2 + 2ax - a^2 = r$, and $\therefore$ $x^2 - 2ax = b - a^2 - r$. Solving this quadratic we find $x = a \pm \sqrt{b - r}$, and here it is evident that r cannot be greater than b; therefore when $r =$ max. we must have $r = b$, $\therefore x = a$.

The same solved without impossible roots.

In the equation $x^2 - 2ax = b - a^2 - r$, let $x = y + a$ $\therefore x^2 - 2ax = y^2 + 2ay + a^2 - 2ay - 2a^2 = y^2 - a^2 =$ $b - a^2 - r$, $\therefore r = b - y^2$ which is evidently a maximum when y^2 or $y = 0$, $\therefore r = b$ and $x = a$ as before.

PROB. (28.) TO FIND SUCH A VALUE FOR x AS WILL MAKE $\frac{x}{1+x^2}$ A MAXIMUM.

Since $\frac{x}{1+x^2} = \text{max.} \therefore \frac{1+x^2}{x} = \text{min.}$ which let $= r$, therefore $x^2 - rx = -1$. Solving this quadratic we find $x = \frac{r}{2} + \sqrt{\frac{r^2}{4} - 1}$, and here it is manifest that r or $\frac{r^2}{4}$ cannot be taken so small as to be less than 1, therefore when $r =$ min. we must have $\frac{r^2}{4} = 1$, $\therefore r = 2$ and $x = \frac{r}{2} = \frac{2}{2} = 1$.

The same solved without impossible roots.

In the equation $x^2 - rx = -1$, let $x = y + \frac{r}{2}$ and therefore $x^2 - rx = y^2 + ry + \frac{r^2}{4} - ry - \frac{r^2}{2} = y^2 - \frac{r^2}{4} = -1$, $\therefore \frac{\ }{4} = y^2 + 1$, which is evidently a minimum when $y = 0$, $\therefore \frac{r^2}{4} = 1$, $\therefore r = 2$ and $x = \frac{r}{2} = \frac{2}{2} = 1$ as before.

PROB. (29.) TO DETERMINE FOR WHAT VALUE OF x THE EXPRESSION $a^4 + b^3x - c^2x^2$ BECOMES A MAXIMUM.

Here $a^4 + b^3x - c^2x^2 = c^2\left(\frac{a^4}{c^2} + \frac{b^3}{c^2}x - x^2\right) = \text{max.}$ or $\frac{a^4}{c^2} + \frac{b^3}{c^2}x - x^2 = \text{max.}$ Now since $\frac{a^4}{c^2}$ is a constant given quantity, we must have $\frac{b^3}{c^2}x - x^2$ also $=$ max. which let $= r$, $\therefore \frac{b^3}{c^2}x - x^2 = r$, or $x^2 - \frac{b^3}{c^2}x = -r$. Solving

this quadratic we find $x = \frac{b^3}{2c^2} + \sqrt{\frac{b^6}{4c^4} - r}$, and here it is evident that r cannot be greater than $\frac{b^6}{4c^4}$ and therefore when r is a maximum we must have $r = \frac{b^6}{4c^4}$ and $x = \frac{b^3}{2c^2}$.

The same solved without impossible roots.

In the expression $x - \frac{b^3}{c^2} x = - r$ let $x = y + \frac{b^3}{2c^2}$ and therefore $x^2 - \frac{b^3}{c^2} x = y^2 + \frac{b^3}{c^2} y + \frac{b^6}{4c^4} - \frac{b^3}{c^2} y - \frac{b^6}{2c^4} = y^2 - \frac{b^6}{4c^4} = - r \therefore r = \frac{b^6}{4c^4} - y^2$ which is evidently a max. when $y = 0, \therefore x = \frac{b^3}{2c^2}$ as before.

PROB. (30.) TO DETERMINE SUCH A VALUE FOR x AS MAY MAKE THE EXPRESSION $a + \sqrt[3]{a^3 - 2a^2x + ax^2}$ A MINIMUM.

Here it is evident that a is a constant given quantity, and consequently $\sqrt[3]{a^3 - 2a^2x + ax^2}$ or its cube $a^3 - 2a^2x + ax^2$ must also be a minimum. Again as a is also a constant given quantity we must have $\frac{a^3 - 2a^2x + ax^2}{a} = a^2 - 2ax + x^2 =$ min. which let $= r \therefore x^2 - 2ax = r - a^2$. Solving this quadratic we find $x = a + \sqrt{r}$, and here it is evident that when $r =$ min. it must be $= 0, \therefore x = a$. This problem may be solved without leaving out any constant given quantity in the following manner, which is more elegant—

Let $a + \sqrt[3]{a^3 - 2a^2x + ax^2} = r, \therefore a^3 - 2a^2x + ax^2 = (r - a)^3 \therefore x^2 - 2ax = \frac{(r - a)^3 - a^3}{a}$. Solving this quad-

ratic we find $x = a + \sqrt{\dfrac{(r-a)^3}{a}}$. Here it is evident that r cannot be taken less than a, because this supposition makes the root impossible: therefore when $r = \text{min.}$ it must be $= a$, $\therefore x = a$ as before.

The same solved without impossible roots.

In the equation $x^2 - 2ax = \dfrac{(r-a)^3 - a^3}{a}$ let $x = y + a$ $\therefore x^2 - 2ax = y^2 + 2ay + a^2 - 2ay - 2a^2 = y^2 - a^2 = \dfrac{(r-a)^3 - a^3}{a}$ $\therefore (r-a)^3 = ay^2$, $\therefore r = a^{\frac{1}{3}}y^{\frac{2}{3}} + a$, which is evidently a minimum when $y = 0$, $\therefore r = a$ and $x = a$ as before.

PROB. (31.) TO FIND THAT NUMBER x WHICH, BEING MULTIPLIED BY THE SQUARE OF ANY GIVEN NUMBER a, AND THE PRODUCT DIVIDED BY THE SQUARE OF THE DIFFERENCE OF a AND x, THE QUOTIENT IS THE GREATEST POSSIBLE.

The product of the square of a and the required number $x = a^2x$, and the square of the difference of a and $x = (a-x)^2$. Therefore the quotient which is to become a maximum is $\dfrac{a^2x}{(a-x)^2}$. Since the reciprocal of a maximum must be a minimum, we must have $\dfrac{(a-x)^2}{a^2x} = \text{min.}$ which let $= r$, $\therefore (a-x)^2 = a^2rx$ or $a^2 - 2ax + x^2 = a^2rx$, $\therefore x^2 - (2a + a^2r)\,x = -a^2$ or $x^2 - a\,(2 + ar)\,x = -a^2$. Solving this quadratic we find,

$$x = \frac{a\,(2+ar)}{2} \pm \sqrt{\frac{4a^2 + 4a^3r + a^4r^2 - 4a^2}{4}}$$

$$= \frac{a\,(2+ar)}{2} + \sqrt{\frac{a^3(4r + ar^2)}{4}}.$$

Here it is evident that when r is a minimum it must be $=$ 0, $\therefore x = \frac{2a}{2} = a$. In this problem impossible roots are not required at all.

The same solved without impossible roots in another way.

In the equation $x^2 - a(2 + ar)x = -a^2$ let $x = y + \frac{a(2+ar)}{2}$ $\therefore x^2 - a(2+ar)x = y^2 + a(2+ar)y + \frac{a^2(2+ar)^2}{4} - a(2+ar)y - \frac{a^2(2+ar)^2}{2} = y^2 - \frac{a^2(2+ar)^2}{4}$ $= -a^2$, $\therefore$ we find $a^2(2+ar)^2 = 4y^2 + 4a^2$, $\therefore r = \frac{2\sqrt{y^2+a^2} - 2a}{a^2}$ which is evidently a minimum when $y = 0$, $\therefore r = \frac{0}{a^2} = 0$, and $x = a$ as before.

PROB. (32.) TO DETERMINE THOSE CONJUGATE DIAMETERS OF AN ELLIPSE WHICH INCLUDE THE GREATEST ANGLE.

Call the principal semi-diameters of the Ellipse a, b, the sought semi-conjugates x and x', and the sine of the angle they include $= y$. Then by conic sections we find $x^2 + x'^2 = a^2 + b^2$ $\therefore x' = \sqrt{a^2 + b^2 - x^2}$ and $xx'y = ab$ $\therefore y = \frac{ab}{xx'}$ and therefore $y = \frac{ab}{x\sqrt{a^2+b^2-x^2}} = \text{min.}$ Here it should be remarked that when we desire to find the greatest value of an angle we may proceed to find the least value of its sine, for the angle is greater and greater as it is more obtuse, and the sine of an angle is the less the greater is its obtuseness. It is for this reason that we have put y, or the value of the sine of the greatest angle, equal to minimum.

Now, omitting the constant given quantity ab, and inverting and squaring the function, we find $(a^2 + b^2)\,x^2 - x^4 =$ max. which let $= r$, and therefore $x^4 - (a^2 + b^2)\,x^2 = -r$. Solving this quadratic we find $x^2 = \frac{a^2 + b^2}{2} \pm \sqrt{\frac{(a^2 + b^2)^2}{4} - r}$ and here it is evident that r cannot be taken greater than $\frac{(a^2 + b^2)^2}{4}$ and therefore when $r =$ max. it must be $= \frac{(a^2 + b^2)^2}{4}$ $\therefore x = \sqrt{\frac{a^2 + b^2}{2}}$. In the solution of this problem that property of the Ellipse is made use of which has not been demonstrated in the Introduction.

The same solved without impossible roots.

In the equation $x^4 - (a^2 + b^2)\,x^2 = -r$, let $x^2 = y + \frac{a^2 + b^2}{2}$, and $\therefore x^4 - (a^2 + b^2)\,x^2 = y^2 + (a^2 + b^2)\,y + \frac{(a^2 + b^2)^2}{4} - (a^2 + b^2)\,y - \frac{(a^2 + b^2)^2}{2} = y^2 - \frac{(a^2 + b^2)^2}{4} = -r$, and therefore $r = \frac{(a^2 + b^2)^2}{4} - y^2$, which is evidently a maximum when $y = 0$, $\therefore x^2 = \frac{a^2 + b^2}{2}$ or $x = \sqrt{\frac{a^2 + b^2}{2}}$ as before.

Now as $y = \frac{ab}{x\sqrt{a^2 + b^2 - x^2}}$ we must have by substitution

$$y = \frac{ab}{\sqrt{\frac{a^2+b^2}{2}}\sqrt{a^2+b^2-\frac{a^2+b^2}{2}}} = \frac{ab}{\sqrt{\frac{a^2+b^2}{2}}\sqrt{\frac{a^2+b^2}{2}}}$$

$$= \frac{ab}{\frac{a^2 + b^2}{2}} = \frac{2ab}{a^2 + b^2}.$$

PROB. (33.) GIVEN THE EQUATION $y^2 - 2mxy + x^2 = a^2$ TO DETERMINE SUCH A VALUE OF x AS WILL MAKE y A MAXIMUM.

From the given equation, in which m is less than unity, we find $x^2 - 2myx = a^2 - y^2$, and solving this quadratic we find $x = my \pm \sqrt{a^2 + (m^2 - 1) y^2} = my \pm \sqrt{a^2 - (1 - m^2) y^2}$. Here it is evident that y cannot be taken so great as to make $(1 - m^2) y^2$ greater than a^2, and therefore when $y =$ max. we must have $(1 - m^2) y^2 = a^2$, $\therefore y = \dfrac{a}{\sqrt{1 - m^2}}$ and $x = \dfrac{ma}{\sqrt{1 - m^2}}$.

The same solved without impossible roots.

In the equation $x^2 - 2myx = a^2 - y^2$ let $x = z + my$, $\therefore x^2 - 2myx = z^2 + 2myz + m^2y^2 - 2myz - 2m^2y^2 = z^2 - m^2y^2 = a^2 - y^2$, $\therefore (1 - m^2) y^2 = a^2 - z^2$, $\therefore y^2 = \dfrac{a^2 - z^2}{1 - m^2}$ which is evidently a maximum when $z = 0$, $\therefore y = \dfrac{a}{\sqrt{1 - m^2}}$ and $x = \dfrac{ma}{\sqrt{1 - m^2}}$ as before.

PROB. (34.) IN A GIVEN CIRCLE TO INSCRIBE THE GREATEST RECTANGLE POSSIBLE. (Fig. 27.)

Let AC be the rectangle, and EF a diameter bisecting BC, $OG = x$ and radius $= a$, then (Euc. III. and II.) $EH = OF$; also, $BO = \sqrt{a^2 - x^2}$ $\therefore$ $BC = 2BO = 2\sqrt{a^2 - x^2}$ and $HO = 2OG = 2x$ $\therefore$ rectangle $AC = 2x \times 2\sqrt{a^2 - x^2}$ or $4x\sqrt{a^2 - x^2} =$ max. and therefore the square of the fourth part of this rectangle, viz. $a^2x^2 - x^4 =$ max. which let $= r$, $\therefore x^4 - a^2x^2 = -r$.

Solving this quadratic, we find $x^2 = \frac{a^2}{2} + \sqrt{\frac{a^4}{4} - r}$, and here it is evident that r cannot be greater than $\frac{a^4}{4}$ and therefore when r is a maximum it must be $= \frac{a^4}{4}$ $\therefore$ $x^2 = \frac{a^2}{2}$ and $x = \frac{a}{\sqrt{2}}$.

The same solved without impossible roots.

In the expression $a^2x^2 - x^4 =$ max. let $x^2 = y + \frac{a^2}{2}$ $\therefore$ $a^2x^2 - x^4 = a^2y + \frac{a^4}{2} - y^2 - a^2y - \frac{a^4}{4} = \frac{a^4}{4} - y^2$ which is evidently a maximum when $y = 0$, and therefore $x^2 = \frac{a^2}{2}$ $\therefore$ $x = \frac{a}{\sqrt{2}}$ as before.

PROB. (35.) THROUGH A GIVEN POINT, WITHIN A GIVEN ANGLE, TO DRAW A STRAIGHT LINE, WHICH SHALL CUT OFF FROM THE ANGULAR SPACE THE SMALLEST TRIANGLE POSSIBLE. (Fig. 28.)

Let P be the given point, A the given angle, and CB the line required. Draw PF and CE perpendicular to AB, and PD parallel to AC: then, since the angle A and the position of P are given, AD, DP and PF are also given.

Let $AD = a$, $DP = b$, $PF = c$, and $AB = x$:

$$\left.\begin{array}{l} \text{then } BD : DP :: BA : AC \\ \text{and } DP : PF :: AC : CE \end{array}\right\} \text{(Euc. 4th, VI.)}$$

From proportion first we find $BD : BA :: DP : AC$, or $AC = \frac{BA \times DP}{BD}$ and from the second proportion $AC = \frac{CE \times DP}{PF}$ $\frac{BA \times DP}{BD} = \frac{CE \times DP}{PF}$, or $\frac{BA}{BD} = \frac{CE}{PF}$ and

hence $BA : BD :: CE : PF$, or $BD : PF :: BA : CE$, or $x - a : c :: x : CE = \frac{cx}{x - a}$ and therefore $ABC = \frac{AB \times CE}{2}$ $\frac{cx^2}{2(x - a)} = \text{min.}$ and $\frac{c}{2}$ is a constant given quantity, therefore $\frac{x^2}{x - a}$ is also a minimum, which let $= r$, $\therefore x^2 = rx - ra$, and $\therefore x^2 - rx = - ra$. Solving this quadratic we find $x = \frac{r}{2} \pm \sqrt{\frac{r^2}{4} - ra} = \frac{r}{2} \pm \sqrt{\frac{r(r - 4a)}{4}}$, and here it is evident that r cannot be less than $4a$, and consequently when r is a minimum we must have $r = 4a \therefore x = \frac{r}{2} = \frac{4a}{2} = 2a$.

Hence, if AB be taken equal to twice AD, the straight line passing through B and P will cut off the smallest triangle possible.

The same solved without impossible roots.

In the equation $x^2 - rx = - ra$ let $x = y + \frac{r}{2}$, $\therefore x^2 - rx = y^2 + ry + \frac{r^2}{4} - ry - \frac{r^2}{2} = y^2 - \frac{r^2}{4} = - ra \therefore \frac{r^2}{4} - ra = y^2$ or $r^2 - 4ra = 4y^2$ or $r = 2a \pm \sqrt{4y^2 + 4a^2}$ which is evidently a minimum when $y = 0$, $\therefore r = 2a + 2a = 4a$ as before, and $x = \frac{r}{2} = \frac{4a}{2} = 2a$.

PROB. (36.) THE RIGHT-ANGLE B OF THE RIGHT-ANGLED TRIANGLE ABC, RESTS UPON THE STRAIGHT LINE DE, TURNING IN ONE PLANE UPON B AS A CENTRE; REQUIRED THE POSITION OF THE TRIANGLE, WHEN THE SUM OF THE PERPENDICULARS AD AND CE IS A MAXIMUM. (Fig. 29.)

Let $AB = a$, $BC = b$, and $AD = x$; then $DB = \sqrt{a^2 - x^2}$; also $AB : BD :: BC : CE$, or $a : \sqrt{a^2 - x^2} :: b : CE = \frac{b}{a}\sqrt{a^2 - x^2}$ and $AD + CE = x + \frac{b}{a}\sqrt{a^2 - x^2} =$ max. which let $= r$, $\therefore \frac{b}{a}\sqrt{a^2 - x^2} = r - x$ and $\frac{a^2b^2 - b^2x^2}{a^2} = \frac{b^2(a^2 - x^2)}{a^2} = r^2 - 2rx + x^2$ or $\frac{a^2 + b^2}{a^2}x^2 - 2rx = b^2 - r^2$ or $x^2 - \frac{2a^2r}{a^2 + b^2}x = \frac{(b^2 - r^2)a^2}{a^2 + b^2}$.

Solving this quadratic we find,

$$x = \frac{a^2r}{a^2 + b^2} \pm \sqrt{\frac{(b^2 - r^2)\,a^2\,(a^2 + b^2) + a^4r^2}{(a^2 + b^2)^2}}$$

$$= \frac{a^2r}{a^2 + b^2} \pm \sqrt{\frac{a^2b^2\,(a^2 + b^2) - a^2b^2r^2}{(a^2 + b^2)^2}}$$

$$= \frac{a^2r}{a^2 + b^2} \pm \sqrt{\frac{a^2b^2\,\{(a^2 + b^2) - r^2\}}{(a^2 + b^2)^2}}.$$

Now it is evident that r^2 cannot be greater than $a^2 + b^2$ and consequently when $r^2 =$ max. it must be equal to $(a^2 + b^2)$ $\therefore r = \sqrt{a^2 + b^2}$ and $x = \frac{a^2\sqrt{a^2 + b^2}}{a^2 + b^2} = \frac{a^2}{\sqrt{a^2 + b^2}} = \frac{AB^2}{AC} =$ a third proportional to AC and AB; which determines the position of the triangle. To find the sum of the perpendiculars, substitute the value of x; then, $CE + AD =$

$$\frac{b}{a}\sqrt{a^2 - \frac{a^4}{a^2 + b^2}} + \frac{a^2}{\sqrt{a^2 + b^2}} = \frac{b}{a} \times \frac{ab}{\sqrt{a^2 + b^2}} + \frac{a^2}{\sqrt{a^2 + b^2}}$$

$= \dfrac{a^2 + b^2}{\sqrt{a^2 + b^2}} = \sqrt{a^2 + b^2} = AC$, $\therefore$ the sum of the perpendiculars, when a maximum $=$ the hypothenuse of the original triangle.

The same solved without impossible roots.

In the equation $x^2 - \dfrac{2a^2r}{a^2 + b^2}x = \dfrac{(b^2 - r^2)a^2}{a^2 + b^2}$ let $x = y + \dfrac{a^2r}{a^2 + b^2}$ $\therefore x^2 - \dfrac{2a^2r}{a^2 + b^2}x = y^2 + \dfrac{2a^2r}{a^2 + b^2}y + \dfrac{a^4r^2}{(a^2 + b^2)^2} - \dfrac{2a^2r}{a^2 + b^2}y - \dfrac{2a^4r^2}{(a^2 + b^2)^2} = y^2 - \dfrac{a^4r^2}{(a^2 + b^2)^2} = \dfrac{(b^2 - r^2)a^2}{a^2 + b^2}$ and therefore $\dfrac{a^2b^2(a^2 + b^2 - r^2)}{(a^2 + b^2)^2} = y^2$, $\therefore r^2 = a^2 + b^2 - \dfrac{y^2(a^2 + b^2)^2}{a^2b^2}$ which is evidently a maximum when $y = 0$, $\therefore r^2 = a^2 + b^2$, $\therefore r = \sqrt{a^2 + b^2}$ and $x = \dfrac{a^2}{\sqrt{a^2 + b^2}}$ as before.

PROB. (37.) TO FIND THE POSITION OF THE SAME TRIANGLE ABC (see last Fig.) WHEN THE SUM OF THE SURFACES OF THE TWO TRIANGLES ADB AND CBE IS A MAXIMUM.

It has already been shown that, if $AB = a$, $BC = b$, and $DA = x$, then $DB = \sqrt{a^2 - x^2}$ and $CE = \dfrac{b}{a}(a^2 - x^2)^{\frac{1}{2}}$;

Now $BA : AD :: CB : BE$, by similar triangles, or, $a : x :: b : BE = \dfrac{b}{a}x$;

$$\therefore ADB + BEC = \frac{AD \times DB}{2} + \frac{BE \times EC}{2}$$

$$= \frac{x}{2}\sqrt{a^2 - x^2} + \frac{b}{2a}x \times \frac{b}{a}\sqrt{a^2 - x^2}$$

$$= \left(\tfrac{1}{2} + \frac{b^2}{2a^2}\right) x\sqrt{a^2 - x^2} = \text{max.}$$

and as $\frac{1}{2} + \frac{b^2}{2a^2}$ is a constant given quantity, we must also have $x\sqrt{a^2 - x^2} =$ max. or its square $a^2x^2 - x^4 =$ max. which let $= r$, $\therefore x^4 - a^2x^2 = -r$. Solving this quadratic we find, $x^2 = \frac{a^2}{2} \pm \sqrt{\frac{a^2}{4} - r}$, and hence it is evident that r cannot be greater than $\frac{a^2}{4}$ and therefore when r is a maximum it must be $= \frac{a^2}{4} \therefore x^2 = \frac{a^2}{2}$ and $x = \frac{a}{\sqrt{2}} = AD$. But, $BD = \sqrt{a^2 - x^2} = \sqrt{a^2 - \frac{a^2}{2}} = \frac{a}{\sqrt{2}} = AD$, $\therefore$ the angle ABD is half a right-angle.

The same solved without impossible roots.

In the equation $a^2x^2 - x^4 = r$ let $x^2 = y + \frac{a^2}{2}$ and therefore $a^2x^2 - x^4 = a^2y + \frac{a^4}{2} - y^2 - a^2y - \frac{a^4}{4} = \frac{a^4}{4} - y^2$ which is evidently a maximum, when $y = 0$, $\therefore x^2 = \frac{a^2}{2}$ and $x = \frac{a}{\sqrt{2}}$ as before.

PROB. (38.) A STRING ABE OF A GIVEN LENGTH, IS FIXED AT A, ONE EXTREMITY OF THE DIAMETER OF A CIRCLE, AND WOUND ROUND PART OF THE ARC AB. THE REMAINDER OF THE LINE, BEING STRETCHED OUT INTO A STRAIGHT LINE AND TERMINATING IN THE DIAMETER PRODUCED; TO FIND THE RADIUS OF THE SEMICIRCLE SO THAT THE AREA BDE, INTERCEPTED BETWEEN THE PRODUCED PART OF THE DIAMETER, THE ARC BD, AND THE STRING, MAY BE A MAXIMUM. (Fig. 30.)

Let $l =$ the length of the string ABE,

$x =$ variable radius BC;

Then from the well-known properties of the circle

Sector $ACB = \dfrac{\text{arc } AB \times BC}{2}$; and semicircle $= \dfrac{px^2}{2}$, where $p =$ circumference of a circle whose diameter is unity. We therefore find $BDE =$ Sector $ACB +$ triangle $CBE -$ semicircle.

The condition that the string is intended to continue on a tangent to the circle is omitted.—Ed.

$$= \frac{AB \times x}{2} + \frac{BE \times x}{2} - \frac{px^2}{2}$$

$$= \frac{(AB + BE)\,x - px^2}{2} = \tfrac{1}{2}\,(lx - px^2)$$

$$= \tfrac{1}{2}p\left(\frac{l}{p}\,x - x^2\right) = \text{max. and since}$$

$\frac{1}{2}p$ is a constant given quantity, we must have also $\dfrac{l}{p}\,x - x^2$ $=$ max. which let $= r$, $\therefore \dfrac{l}{p}\,x - x^2 = r$, and $x^2 - \dfrac{l}{p}\,x =$ $- r$. Solving this quadratic we find $x = \dfrac{l}{2p} \pm \sqrt{\dfrac{l^2}{4p^2} - r}$, and it is manifest that r cannot be greater than $\dfrac{l^2}{4p^2}$, and consequently when $r =$ max. we must have $r = \dfrac{l^2}{4p^2}$ $\therefore x =$ $\dfrac{l}{2p}$ or radius $= \dfrac{l}{2p}$ and therefore $l = 2p \times$ radius $= p \times$ diameter $=$ circumference; and hence it appears that the radius is such that, if the circle were completed, its circumference would be equal to the length of the string.

The same solved without impossible roots.

In the expression $\dfrac{l}{p}\,x - x^2 =$ max. let $x = y + \dfrac{l}{2p}$ and therefore $\dfrac{l}{p}\,x - x^2 = \dfrac{l}{p}\,x + \dfrac{l^2}{2p^2} - y^2 - \dfrac{l}{p}\,x - \dfrac{l^2}{4p^2} =$ $\dfrac{l^2}{4p^2} - y^2$ which is evidently a maximum, when $y = 0$ and therefore $x = \dfrac{l}{2p}$ as before.

PROB. (39.) GIVEN A POINT A, IN THE RADIUS BC, OF THE SEMICIRCLE DEB; TO FIND THE POINT E AT WHICH, IF A TANGENT EG BE DRAWN, THE ANGLE AEG, FORMED BY AE AND EG, SHALL BE A MINIMUM. (Fig. 31.)

Let C be the centre, $CA = a$, $AE = x$, $CE = b$, the angle $CEA = \phi$.

Then, since CEG is a right angle, and therefore a constant quantity, it follows that, when AEG is a minimum, AEC is a maximum; and the problem resolves itself into the determination of E when ϕ is a maximum. Now, by prop. 13th of the 2nd book of Euclid, and by principles of Trigonometry, we find $a^2 = b^2 + x^2 - 2bx \cos\phi$, and therefore $\cos\phi = \dfrac{b^2 + x^2 - a^2}{2bx}$. But ϕ is always less than a right angle; hence when ϕ is a maximum, $\cos\phi$ will be a minimum;

$\therefore \dfrac{b^2 + x^2 - a^2}{2bx} = \text{min. which let} = r,$

$\therefore b^2 + x^2 - a^2 = 2bxr$ or $x^2 - 2brx = a^2 - b^2$. Solving this quadratic we find $x = br \pm \sqrt{b^2r^2 - b^2 + a^2}$, and it is evident, by inspection of the diagram, that CB is greater than CA $\therefore b^2 > a^2$ and $a^2 - b^2 =$ a negative quantity, which let $= -P^2$ $\therefore x = br \pm \sqrt{b^2r^2 - P^2}$. Now it is clear that r cannot be taken so small as to make b^2r^2 less than P^2, and therefore when $r = $ min. we must have $b^2r^2 = P^2 = b^2 - a^2$ and $r = \sqrt{\dfrac{b^2 - a^2}{b^2}}$ and $x = br = b\sqrt{\dfrac{b^2 - a^2}{b^2}} = \sqrt{b^2 - a^2}$ or $b^2 = a^2 + x^2$ or $CE^2 = CA^2 + AE^2$, and hence it appears that CAE is a right angle.

The same solved without impossible roots.

In the equation $x^2 - 2brx = a^2 - b^2$ let $x = y + br$ $\therefore x^2 - 2brx = y^2 + 2bry + b^2r^2 - 2bry - 2b^2r^2 = y^2 - b^2r^2 = a^2 - b^2 = - (b^2 - a^2) \therefore r^2 = \frac{y^2 + b^2 - a^2}{b^2}$ which is evidently a minimum when $y = 0$, $\therefore r^2 = \frac{b^2 - a^2}{b^2} \therefore r = \sqrt{\frac{b^2 - a^2}{b^2}} = \frac{1}{b}\sqrt{b^2 - a^2}$ and $x = br = \sqrt{b^2 - a^2}$ as before.

PROB. (40.) TO FIND A POINT D, IN THE SEMICIRCLE ADB, SUCH THAT THE SUM OF THE DISTANCES $AD + DP$ MAY BE A MAXIMUM; P BEING A GIVEN POINT IN THE RADIUS BC. (Fig. 32.)

Let D be the required point: draw DE perpendicular to AB; also, let $AC = a$, $AE = x$, $CP = b$. Then by prop. 35, 3rd book of Euclid, we find, $DE^2 = 2ax - x^2$; therefore $PD = \sqrt{DE^2 + EP^2} = \sqrt{2ax - x^2 + (a + b - x)^2} = \sqrt{(a + b)^2 - 2bx}$. Now by prop. 8 of the 6th book of Euclid $AD = \sqrt{AB \times AE} = \sqrt{2ax} \therefore AD + PD = \sqrt{2ax} + \sqrt{(a + b)^2 - 2bx} =$ maximum. Let $\sqrt{2ax} = y \therefore x = \frac{y^2}{2a}$ and $2bx = \frac{by^2}{a}$ and therefore $y + \sqrt{(a + b)^2 - \frac{by^2}{a}} =$ max. which let $= r$, and consequently $(a + b)^2 - \frac{by^2}{a} = r^2 - 2ry + y^2 \therefore \frac{a + b}{a} y^2 - 2ry = (a + b)^2 - r^2$, and $\therefore y^2 - \frac{2ar}{a + b} y = \frac{a(a + b)^2 - ar^2}{a + b}$.

Solving this quadratic we find $y = \frac{ar}{a + b} \pm \sqrt{\frac{a(a + b)^3 - abr^2}{(a + b)^2}}$ and hence it is evident that r cannot be so great as to make

abr^2 greater than $a(a+b)^3$, and therefore when r is a maximum we must have $abr^2 = a(a+b)^3$ $\therefore r = \sqrt{\dfrac{(a+b)^3}{b}}$,

and $\therefore y = \dfrac{ar}{a+b} = \sqrt{\dfrac{a^2(a+b)}{b}}$ and $x = \dfrac{y^2}{2a} = \dfrac{a(a+b)}{2b}$; converting this into an analogy, we have $2b : a :: a+b : x$.

From this it appears that if from AB we cut off AE, a fourth proportional to $2CP$, AC and AP, and through E draw ED perpendicular to AB, meeting the circumference in D, then D is the point required. Since x or $AE = \dfrac{a(a+b)}{2b}$, it follows that, as b decreases, x must increase, and that when $b = 0$, $x = \dfrac{a^2}{0} =$ infinity. This is no doubt a fair and legitimate conclusion, when the value of x is viewed as an abstract formula; it is inconsistent, however, with the nature of the problem before us, in which we perceive that x, so far from admitting of indefinite increase, can never exceed the diameter AB or $2a$. This limit above which x cannot ascend, will naturally fix a corresponding limit, below which b cannot descend; to reach this we have merely to substitute for x its greatest value $2a$ in the equation $x = \dfrac{a(a+b)}{2b}$; the resolution of which will give the minimum value required; thus, $2a = \dfrac{a(a+b)}{2b}$ $\therefore b = \dfrac{a}{3}$; that is, the conditions of possibility fix P between B and another point distant from it by $\frac{2}{3}$ the radius of the circle.

The same solved without impossible roots.

In the equation $y^2 - \dfrac{2ar}{a+b}y = \dfrac{a(a+b)^2 - ar^2}{a+b}$ let $y = z + \dfrac{ar}{a+b}$ $\therefore y^2 - \dfrac{2ar}{a+b}y = z^2 + \dfrac{2ar}{a+b}z + \dfrac{a^2r^2}{(a+b)^2} -$

$$\frac{2ar}{a+b}z - \frac{2a^2r^2}{(a+b)^2} = z^2 - \frac{a^2r^2}{(a+b)^2} = \frac{a(a+b)^2 - ar^2}{a+b}$$

and therefore $r^2 = \frac{a(a+b)^3 - (a+b)^2 z^2}{ab}$ which is evidently a maximum when $z = 0$, $\therefore r^2 = \frac{(a+b)^2}{b}$ and $y = \frac{ar}{a+b} = \sqrt{\frac{a^2(a+b)}{b}}$ and $x = \frac{y^2}{2a} = \frac{a(a+b)}{2b}$ as before.

PROB. (41.) OF ALL THE CONES WHICH CAN CIRCUMSCRIBE A GIVEN SPHERE, TO FIND THAT WHICH HAS THE LEAST POSSIBLE SOLIDITY. (Fig. 33.)

Let *D mn* and *AEB* be the circular, and triangular sections of the given sphere, and the required cone the solidity of which is to become a minimum.

Let $CD = a =$ radius of the sphere.

$CE = x$ and $Am = y =$ radius of the base of the cone. It is evident that the angle *EDC* is a right angle, and consequently the triangle *EDC* is equiangular and similar to the triangle *EmA* $\therefore Em : mA :: ED : DC$ or $x + a : y :: \sqrt{x^2 - a^2} : a$, and therefore $y = \frac{a(x+a)}{\sqrt{x^2 - a^2}}$ $\therefore$ the area of the circle, which is the base of the cone $= py^2$ (where $p =$ circumference of the circle whose diameter is unity) $= \frac{pa^2(x+a)^2}{x^2 - a^2} = pa^2 \times \frac{(a+x)^2}{(a+x)(x-a)}$, and therefore the solid contents of the required cone $= pa^2 \times \frac{(x+a)^2}{(x+a)(x-a)} \times \frac{x+a}{3} = \frac{pa^2}{3} \times \frac{(x+a)^2}{x-a} =$ min. Let $y = x - a$ $\therefore x + a = y + 2a$, and we therefore find $\frac{pa^2}{3} \times \frac{(x+a)^2}{x-a} = \frac{pa^2}{3}$

$\times \dfrac{(y + 2a)^2}{y}$ = min. and since $\dfrac{pa^2}{3}$ is a constant given quantity, we must also have $\dfrac{(y + 2a)^2}{y}$ = min. which let $= r$, and therefore $y^2 + 4ay + 4a^2 = ry \therefore y^2 + (4a - r)\, y = - 4a^2$. Solving this quadratic we find $y = - \dfrac{4a - r}{2} \pm \sqrt{\dfrac{(4a - r)^2}{4} - 4a^2} = - \dfrac{4a - r}{2} \pm \sqrt{\dfrac{r(r - 8a)}{4}}$ and here it is evident that r cannot be less than $8a$, and therefore when r is a minimum, we must have $r = 8a$, and $\therefore y = - \dfrac{4a - r}{2} = \dfrac{4a}{2} = 2a$ and $x = y + a = 3a \therefore Em = x + a = 4a$ = twice the diameter of the given sphere. Hence it appears that the altitude of the smallest cone which can be circumscribed about a given sphere, is equal to twice the diameter of the sphere.

The same solved without impossible roots.

In the equation $y^2 + (4a - r)\, y = - 4a^2$ let $y = z - \dfrac{4a - r}{2} \therefore y^2 + (4a - r)\, y = z^2 - (4a - r)\, z + \dfrac{(4a - r)^2}{2} + (4a - r)^2 z - \dfrac{(4a - r)^2}{2} = z^2 - \dfrac{(4a - r)^2}{4} = - 4a^2$ and therefore $4z^2 + 16a^2 = (4a - r)^2 = (r - 4a)^2 \therefore r = 4a + \sqrt{4z^2 + 16a^2}$; here it is manifest that when r is a minimum we must have $z = 0$, and therefore $r = 8a \therefore y = - \dfrac{4a - r}{2} = 2a$ and $x = y + a = 3a$ and $Em = x + a = 4a$ as before.*

* Here y has been used in two different senses, but not so as to produce confusion.—Ed.

PROB. (42.) TO FIND THAT NUMBER WHICH BEING ADDED TO ITS RECIPROCAL THE SUM IS THE LEAST POSSIBLE.

Let $x =$ number required and $\frac{1}{x} =$ its reciprocal.

Now by the conditions of the problem we have $x + \frac{1}{x} =$ min. or $\frac{x^2+1}{x} =$ min. which let $= r$, $\therefore x^2 - rx = -1$. Solving this quadratic we find $x = \frac{r}{2} \pm \sqrt{\frac{r^2}{4} - 1}$, and hence it is evident that r cannot be taken so small as to make $\frac{r^2}{4}$ less than 1, and therefore when $r =$ min. we must have $\frac{r^2}{4} = 1$, $\therefore r = 2$ and $x = \frac{r}{2} = 1$.

The same solved without impossible roots.

In the equation $x^2 - rx = -1$, let $x = y + \frac{r}{2}$ and therefore $x^2 - rx = y^2 + ry + \frac{r^2}{4} - ry - \frac{r^2}{2} = y^2 - \frac{r^2}{4} = -1$, $\therefore r^2 = 4y^2 + 4$ which is evidently a minimum when $y = 0$, $\therefore r = 2$ and $x = \frac{r}{2} = 1$ as before.

PROB. (43.) *AC* AND *BD* BEING PARALLEL, IT IS REQUIRED TO DRAW FROM *C* A LINE *CXY* SUCH THAT THE SUM OF THE TRIANGLES *ACX* AND *BXY* SHALL BE A MINIMUM. (Fig. 34.)

If $AC = a$, $AB = b$, $AX = x$, it is easily seen that the area of the triangle ACX is proportional to ax, and that of BXY to $\frac{a(b-x)^2}{x}$, so that we have $a\left\{x + \frac{(b-x)^2}{x}\right\} =$

minimum, and therefore $x + \frac{(b-x)^2}{x} =$ min. which let $= r$, $\therefore\ x^2 + b^2 - 2bx + x^2 = rx$, $\therefore\ x^2 - \frac{2b+r}{2}x = -\frac{b^2}{2}$. Solving this quadratic we find

$$x = \frac{2b+r}{4} \pm \sqrt{\frac{4b^2 + 4br + r^2 - 8b^2}{16}} = \frac{2b+r}{4}$$
$$\pm \sqrt{\frac{(4b+r)r - 4b^2}{16}}$$

and here it is evident that r cannot be taken so small as to make $(4b + r)\, r$ less than $4b^2$, and therefore when $r =$ min. we must have $r^2 + 4br = 4b^2\ \therefore\ r = \sqrt{8b^2} - 2b$ and $x = \frac{2b+r}{4} = \frac{2b - 2b + \sqrt{8b^2}}{4} = \frac{2b\sqrt{2}}{4} = \frac{b}{\sqrt{2}}$ which determines the line CXY.

The same solved without impossible roots.

In the equation $x^2 - \frac{2b+r}{2}x = -\frac{b^2}{2}$ let $x = y + \frac{2b+r}{4}$ $\therefore\ x^2 - \frac{2b+r}{2}x = y^2 + \frac{2b+r}{2}y + \frac{(2b+r)^2}{16} - \frac{2b+r}{2}$ $y - \frac{(2b+r)^2}{8} = y^2 - \frac{(2b+r)^2}{16} = -\frac{b^2}{2}$ and therefore $r = \sqrt{16y^2 + 8b^2} - 2b$, which is evidently a minimum when $y = 0$, $\therefore\ r = \sqrt{8b^2} - 2b$ and $x = \frac{2b+r}{4} = \frac{b}{\sqrt{2}}$ as before.

PROB. (44.) TO FIND THE HEIGHT ABOVE THE GIVEN POINT A FROM WHENCE AN ELASTIC BALL MUST BE SUFFERED TO DESCEND FREELY BY GRAVITY, SO THAT, AFTER STRIKING THE HARD PLANE AT B, IT MAY BE REFLECTED BACK AGAIN TO THE POINT A, IN THE LEAST TIME POSSIBLE, FROM THE INSTANT OF DROPPING IT. (Fig. 35.)

Let C be the point required, and put $AC = x$, and $AB = a$; then the spaces of falling bodies, by the force of gravity being as the squares of the times, we find $CB = gt^2$ and $CA = gt'^2$, where $g = 16$ feet nearly, and consequently $t = \frac{1}{\sqrt{g}}\sqrt{CB}$ and $t' = \frac{1}{\sqrt{g}}\sqrt{CA}$, and therefore $t - t' = \frac{1}{\sqrt{g}}\sqrt{CB} - \frac{1}{\sqrt{g}}\sqrt{CA} = \frac{1}{\sqrt{g}}\sqrt{a+x} - \frac{1}{\sqrt{g}}\sqrt{x}$ = the time down AB, or the time of rising from B to A again: hence the whole time of falling through CB and returning to A is $\frac{1}{\sqrt{g}}\sqrt{a+x} - \frac{1}{\sqrt{g}}\sqrt{x} + \frac{1}{\sqrt{g}}\sqrt{a+x} = \frac{1}{\sqrt{g}}(2\sqrt{a+x} - \sqrt{x})$ = min. and as $\frac{1}{\sqrt{g}}$ is a constant given quantity, we must have $2\sqrt{a+x} - \sqrt{x}$ = min. which let $= r$ $\therefore 2\sqrt{a+x} = r + \sqrt{x}$. Now let $\sqrt{x} = y$ $\therefore x = y^2$ $\therefore 2\sqrt{a+y^2} = r + y$ and squaring both sides of the equation we find $4a + 4y^2 = r^2 + 2ry + y^2$ and $y^2 - \frac{2r}{3}y = \frac{r^2 - 4a}{3}$. Solving this quadratic we find $y = \frac{r}{3} \pm \sqrt{\frac{4r^2 - 12a}{9}}$, and here it is evident that r cannot be taken so small as to make $4r^2$ less than $12a$, and therefore when r = min. we must have $4r^2 = 12a$, and therefore $r = \sqrt{3a} = \sqrt{a}\sqrt{3}$ and $y = \frac{\sqrt{a}\sqrt{3}}{3} = \sqrt{\frac{a}{3}}$ and $x = y^2 = \frac{a}{3}$, that is $AC = \frac{1}{3}AB$.

The same solved without impossible roots.

In the equation $y^2 - \frac{2r}{3}y = \frac{r^2 - 4a}{3}$ let $y = z + \frac{r}{3}$, therefore $y^2 - \frac{2r}{3}y = z^2 + \frac{2r}{3}z + \frac{r^2}{9} - \frac{2r}{3}z - \frac{2r^2}{9} =$ $z^2 - \frac{r^2}{9} = \frac{r^2 - 4a}{3} = \frac{3r^2 - 12a}{9}$ $\therefore z^2 = \frac{r^2}{9} + \frac{3r^2}{9} - \frac{12a}{9}$ and $4r^2 = 9z^2 + 12a$ $\therefore r = \sqrt{\frac{9z^2 + 12a}{4}}$ which is evidently a minimum when $z = 0$, $\therefore r = \sqrt{3}\sqrt{a}$ and $y = \frac{r}{3} = \sqrt{\frac{a}{3}}$ $\therefore x = y^2 = \frac{a}{3}$ as before.

PROB. (45.) GIVEN THE HEIGHT OF AN INCLINED PLANE; TO FIND ITS LENGTH, SO THAT A GIVEN POWER ACTING ON A GIVEN WEIGHT, IN A DIRECTION PARALLEL TO THE GIVEN PLANE, MAY DRAW IT UP IN THE LEAST TIME POSSIBLE.

Let a denote the height of the plane, x its length, p the power, and w the weight. Now the tendency down the plane is $= gw$ sin. of the angle made by the length with the base of the plane $= gw\frac{a}{x} = \frac{gaw}{x}$, where g = force of gravity = $32\frac{1}{6}$ feet, and the tendency up the plane $= gp$ $\therefore$ the whole motive power up the plane $= gp - \frac{gwa}{x} = \frac{(px - aw)g}{x}$; but the mass resisting this motion is $p + w$, therefore the accelerating force for raising the weight upon the plane is equal to $\frac{(px - aw)g}{(p + w)x}$. Now the space ascended =

$x = ft^2 = \frac{(px - aw)g}{(p + w)x}\,t^2$ where f = force $\therefore t^2 = \frac{(p + w)x^2}{(px - aw)g}$ = min. and $\therefore \frac{x^2}{px - aw}$ = min. which let $= r$, and therefore $x^2 = prx - awr \therefore x^2 - prx = -awr$. Solving this quadratic we find, $x = \frac{pr}{2} \pm \sqrt{\frac{p^2r^2 - 4awr}{4}} = \frac{pr}{2} \pm \sqrt{\frac{r(p^2r - 4aw)}{4}}$, and hence it is evident that r cannot be taken so small as will make p^2r less than $4aw$, and therefore when r = min. we must have $p^2r = 4aw$ and $r = \frac{4aw}{p^2} \therefore x = \frac{pr}{2} = \frac{2aw}{p}$ and $\therefore p : w :: 2a : x ::$ double the height of the plane : its length.

The same solved without impossible roots.

In the equation $x^2 - prx = -awr$ let $x = y + \frac{pr}{2}$, therefore $x^2 - prx = y^2 + pry + \frac{p^2r^2}{4} - pry - \frac{p^2r^2}{2} = y^2 - \frac{p^2r^2}{4} = -awr \therefore p^2r^2 - 4awr = 4y^2$ or $r^2 - \frac{4aw}{p^2}r = \frac{4y^2}{p^2}$ and therefore $r = \frac{2aw}{p^2} + \sqrt{\frac{4a^2w^2 + 4y^2p^2}{p^4}}$ which is evidently a minimum when $y = 0$, and $\therefore r = \frac{4aw}{p^2}$ and $x = \frac{pr}{2} = \frac{2aw}{p}$ as before.

*** Here p and w are *masses*, not *weights*, as stated; and $\frac{1}{2}f$ should have been used instead of f.—Ed.

PROB. (46.) A LARGE VESSEL OF 10 FEET, OR ANY OTHER GIVEN DEPTH, AND OF ANY SHAPE, BEING KEPT CONSTANTLY FULL OF WATER, BY MEANS OF A SUPPLYING COCK, AT THE TOP; IT IS PROPOSED TO ASSIGN THE PLACE WHERE A SMALL HOLE MUST BE MADE IN THE SIDE OF IT, SO THAT THE WATER MAY SPOUT THROUGH IT TO THE GREATEST DISTANCE ON THE PLANE OF THE BASE. (Fig. 36.)

Let AB denote the height or side of the vessel; D the required hole in the side, from which the water spouts, in the parabolic curve DG, to the greatest distance BG, on the horizontal plane.

It is evident that the velocity of the water descending from A to D with which it must spout out in the horizontal direction must be expressed by the equation $v = \sqrt{2gs} = \sqrt{2} \times \sqrt{s}\sqrt{g} = \sqrt{2} \times \sqrt{AD} \times \sqrt{g}$ .... (1)

It is also evident that the time t in which the water spouting out from the hole at D must reach the ground, must be the same in which it may descend from D to B and $t^2 = \dfrac{DB}{\frac{1}{2}g} = \dfrac{2DB}{g} \therefore t = \dfrac{\sqrt{2} \times \sqrt{DB}}{\sqrt{g}}$ .... (2)

Multiplying the equations (1) and (2) we find $tv =$ horizontal space $GB = 2\sqrt{AD \cdot DB} =$ maximum, and supposing $AB = a$ and $AD = x$ we find $2\sqrt{x(a-x)} = 2\sqrt{ax - x^2} =$ max. and $\therefore ax - x^2 =$ max. which let $= r$, and therefore $x^2 - ax = -r$. Solving this quadratic we find, $x = \dfrac{a}{2} \pm \sqrt{\dfrac{a^2}{4} - r}$, and hence it is manifest that r cannot be greater than $\dfrac{a^2}{4}$, and consequently when $r =$ max. we must have

$\frac{a^2}{4} = r$ and $x = \frac{a}{2}$. So that the hole must be in the middle between the top and the bottom.

The same solved without impossible roots.

In the equation $ax - x^2$ let $x = y + \frac{a}{2}$, and therefore we find $ax - x^2 = ay + \frac{a^2}{2} - y^2 - ay - \frac{a^2}{4} = \frac{a^2}{4} - y^2$ which is evidently a maximum when $y = 0$, $\therefore x = \frac{a}{2}$ as before.

PROB. (47.) IF THE SAME VESSEL, AS IN PROBLEM 46, STAND ON HIGH, IT IS PROPOSED TO DETERMINE WHERE THE SMALL HOLE MUST BE MADE, SO AS TO SPOUT FARTHEST ON THE SAID PLANE. (Fig. 37.)

Let the annexed figure represent the vessel as before, and bG the greatest distance spouted by the fluid DG, on the plane bG. Here, as before, $bG = 2\sqrt{AD \cdot Db} = 2\sqrt{x(c-x)}$ $= 2\sqrt{cx - x^2}$, by putting $Ab = c$, and $AD = x$. So that $2\sqrt{cx - x^2}$ or $cx - x^2$ must be a maximum, which let $= r$, and therefore $x^2 - cx = -r$. Solving this quadratic we find, $x = \frac{c}{2} \pm \sqrt{\frac{c^2}{4} - r}$, and hence it is evident that r cannot be greater than $\frac{c^2}{4}$, and consequently when r is a maximum we must have $r = \frac{c^2}{4}$ and therefore $x = \frac{c}{2}$. So that the hole D must be made in the middle, between the top of the vessel and the given plane, that the water may spout farthest.

The same may be solved without impossible roots, as problem (46.)

PROB. (48.) TO DIVIDE A NUMBER a INTO TWO SUCH PARTS THAT IF THE SQUARE OF ONE OF THESE BE SUBTRACTED FROM THEIR PRODUCT, THE REMAINDER IS THE GREATEST POSSIBLE.

Let $x =$ one of the parts, and therefore $a - x =$ the other part, $\therefore ax - x^2 =$ product of the two parts, and $x^2 =$ square of one of them, and therefore $ax - x^2 - x^2 = ax - 2x^2 =$ max. which let $= r \therefore x^2 - \frac{a}{2}x = -\frac{r}{2}$. Solving this quadratic we find $x = \frac{a}{4} \pm \sqrt{\frac{a^2}{16} - \frac{r}{2}} = \frac{a}{4} \pm \sqrt{\frac{a^2 - 8r}{16}}$, and hence it is manifest that r cannot be taken so great as to make $8r$ greater than a^2, and consequently when r is a maximum we must have $a^2 = 8r$, and therefore $x = \frac{a}{4}$

The same solved without impossible roots.

In the expression $ax - 2x^2$ or its half $\left(\frac{a}{2}x - x^2\right)$ which is made a maximum, let $x = y + \frac{a}{4}$ and therefore $\frac{a}{2}x - x^2 = \frac{a}{2}y + \frac{a^2}{8} - y^2 - \frac{a}{2}y - \frac{a^2}{16} = \frac{a^2}{16} - y^2$ which is evidently a maximum when $y = 0$, and $\therefore x = \frac{a}{4}$ as before.

PROB. (49.) TO FIND THE POINT IN THE LINE JOINING THE CENTRES OF TWO SPHERES FROM WHICH THE GREATEST PORTION OF SPHERICAL SURFACE IS VISIBLE. (Fig. 38.)

Let npA and Dgs be two great circles of the two spheres in the same plane, AD their common tangent, and C and m

their common centres. Also let $Cm = c$, $Cv = a$, $wm = b$, and $CB = x$. Now by similar triangles (prop. 8th of 6th book of Euclid) we have $CB : CA :: CA : Cb$, or $x : a :: a : Cb = \frac{a^2}{x}$ $\therefore$ $bv = Cv - Cb = a - \frac{a^2}{x}$ and $dw = mw - dm = b - \frac{b^2}{c - x}$. The surface of the spherical segment whose height is $bv = 2pa \times bv$ (p = circumference of a circle whose diameter is unity) $= 2p\left(a^2 - \frac{a^3}{x}\right)$ and the surface of the spherical segment whose height is $wd = 2pb \times md$ $= 2p\left(b^2 - \frac{b^3}{c - x}\right)$ and therefore the sum of the surfaces of portions of the two given spheres $= 2p\left(a^2 - \frac{a^3}{x}\right) +$ $2p\left(b^2 - \frac{b^3}{c - x}\right) = 2p\left(a^2 + b^2 - \frac{a^3}{x} - \frac{b^3}{c - x}\right) =$ max. and since $2p$ is a constant given quantity, we must also have $a^2 + b^2 - \left(\frac{a^3}{x} + \frac{b^3}{c - x}\right) =$ max. which let $= q$, therefore $\frac{a^3}{x} + \frac{b^3}{c - x} = a^2 + b^2 - q$. Here it is evident that when $q =$ max. $a^2 + b^2 - q$ must be a minimum, which let $= r$, and therefore $\frac{a^3}{x} + \frac{b^3}{c - x} = \frac{ca^3 + (b^3 - a^3)x}{cx - x^2} =$ min. which let $= r$ $\therefore$ $\frac{ca^3 + (b^3 - a^3)x}{cx - x^2} = r$. Now let $x = \frac{c}{y + 1}$, therefore $\frac{ca^3 + (b^3 - a^3)x}{cx - x^2} = \frac{ca^3 + (b^3 - a^3)\frac{c}{y + 1}}{\frac{c^2}{y + 1} - \frac{c^2}{(y + 1)^2}} =$

$$\frac{a^3(y+1)^2 + (b^3 - a^3)(y+1)}{cy} = \frac{a^3y^2 + 2a^3y + a^3 + (b^3 - a^3)y + b^3 - a^3}{cy}$$

$= \frac{b^3 - a^3 + 2a^3}{c} + \frac{a^3y^2 + b^3}{cy} =$ min. and since $\frac{b^3 - a^3 + 2a^3}{c}$

is a constant given quantity, we must have $\frac{a^3y^2 + b^3}{cy} = \text{min.}$ which let $= r$, $\therefore a^3y^2 + b^3 = cry$ and $y^2 - \frac{cr}{a^3}y = -\frac{b^3}{a^3}$.

Solving this quadratic we find $y = \frac{cr}{2a^3} \pm \sqrt{\frac{c^2r^2 - 4a^3b^3}{4a^6}}$ and here it is evident that r cannot be taken so small as to make c^2r^2 less than $4a^3b^3$, and therefore when $r = \text{min.}$ we must have $c^2r^2 = 4a^3b^3$, $\therefore r = \frac{2a^{\frac{3}{2}}b^{\frac{3}{2}}}{c}$ and $y = \frac{cr}{2a^3} = \frac{b^{\frac{3}{2}}}{a^{\frac{3}{2}}}$;

therefore $x = \frac{c}{y+1} = \frac{c}{\frac{b^{\frac{3}{2}}}{a^{\frac{3}{2}}}+1} = \frac{ca^{\frac{3}{2}}}{a^{\frac{3}{2}} + b^{\frac{3}{2}}}$.

The same solved without impossible roots.

In the equation $y^2 - \frac{cr}{a^3}y = -\frac{b^3}{a^3}$ let $y = z + \frac{cr}{2a^3}$

$\therefore y^2 - \frac{cr}{a^3}y = z^2 + \frac{cr}{a^3}z + \frac{c^2r^2}{4a^6} - \frac{cr}{a^3}z - \frac{c^2r^2}{2a^6} = z^2 - \frac{c^2r^2}{4a^6} = -\frac{b^3}{a^3}$ $\therefore r^2 = \frac{4a^6z^2 + 4a^3b^3}{c^2}$ which is evidently a minimum when $x = 0$ $\therefore r = \frac{2a^{\frac{3}{2}}b^{\frac{3}{2}}}{c}$ and $y = \frac{cr}{2a^3} = \frac{b^{\frac{3}{2}}}{a^{\frac{3}{2}}}$

$\therefore x = \frac{c}{y+1} = \frac{ca^{\frac{3}{2}}}{a^{\frac{3}{2}} + b^{\frac{3}{2}}}$ as before.

*** Inaccurate description of the figure: *BD* and *BA* are not in the same straight line.—Ed.

PROB. (50.) TO FIND THE VALUE OF THE ANGLE x WHEN m SIN. $(x - a)$ COS. x = MAXIMUM.

It is evident that m being a constant given quantity, we must have sin. $(x - a)$ cos. x = sin. x cos. a cos. x − sin. a cos.^{2} x = max.

Now let cos. $x = y$, cos. $a = b$, and sin. $a = \sqrt{1 - b^2} = c$

$\therefore by\sqrt{1-y^2} - cy^2 = c\left(\frac{b}{c}\sqrt{y^2-y^4} - y^2\right) = \text{max. or}$
$\frac{b}{c}\sqrt{y^2-y^4} - y^2 = \text{max.}$ which let $= r$, and therefore $\frac{b^2}{c^2}$
$y^2 - \frac{b^2}{c^2}y^4 = y^4 + 2y^2 r + r^2$ or $\frac{b^2+c^2}{c^2}y^4 - \frac{b^2-2rc^2}{c^2}$
$y^2 = -r^2$. But $b^2 + c^2 = b^2 + 1 - b^2 = 1$, and therefore
$\frac{1}{c^2}y^4 - \frac{b^2-2rc^2}{c^2}y^2 = -r^2 \therefore y^4 - (b^2-2rc^2)\,y^2 = -r^2c^2.$
Solving this quadratic we find

$$y^2 = \frac{b^2-2rc^2}{2} \pm \sqrt{\frac{b^4 - 4b^2c^2r - 4r^2c^2\,(1-c^2)}{4}} \quad \ldots(1)$$

Now c is the sine of a given angle, $\therefore c^2$ must be less than unity, and consequently $1 - c^2$ must be positive, and hence it appears that, excepting b^4, all the rest of the terms in the numerator of the fraction under square root are negative, and for this reason we cannot take for r so great a value as will make $4b^2c^2r + 4r^2c^2\,(1-c^2)$ greater than b^4; hence when $r = \text{max.}$ we must have
$4r^2c^2\,(1-c^2) + 4b^2c^2r = b^4$, and from this quadratic we find $r^2 + \frac{b^2}{1-c^2}r = \frac{b^4}{4c^2\,(1-c^2)}$ and therefore $r =$

$$\sqrt{\frac{b^4c^2 + b^4 - b^4c^2}{4c^2(1-c^2)^2}} - \frac{b^2}{2(1-c^2)} = \frac{b^2}{2c(1-c^2)} - \frac{b^2}{2(1-c^2)}$$

$= \frac{b^2(1-c)}{2c(1-c^2)} = \frac{b^2}{2c(1+c)}$. Now from equation (1) we find

$$y^2 = \frac{b^2-2rc^2}{2} = \frac{b^2 - \frac{b^2c}{1+c}}{2} = \frac{b^2}{2(1+c)} = \frac{\cos.^2 a}{2(1+\sin.a)}$$

$$= \frac{1-\sin.^2 a}{2(1+\sin.a)} = \frac{1-\sin.a}{2} = \frac{\cos.^2\frac{a}{2} - 2\sin.\frac{a}{2}\cos.\frac{a}{2} + \sin.^2\frac{a}{2}}{2}$$

$$= \left(\frac{\cos.\frac{a}{2} - \sin.\frac{a}{2}}{\sqrt{2}}\right)^2 \therefore y = \cos.\frac{a}{2} \times \frac{1}{\sqrt{2}} - \sin.\frac{a}{2} \times \frac{1}{\sqrt{2}}$$

$= \cos.\frac{a}{2}\cos.45° - \sin.\frac{a}{2}\sin.45° = \cos.\left(45 + \frac{a}{2}\right)$ but $y = \cos.x$, $\therefore x = \frac{a}{2} + 45°$.

The same solved without impossible roots.

In the equation $y^4 - (b^2 - 2rc^2)\,y^2 = -c^2r^2$ let $y^2 = z + \frac{b^2 - 2rc^2}{2}$ $\therefore y^4 - (b^2 - 2rc^2)\,y^2 = z^2 + (b^2 - 2rc^2)\,z + \frac{(b^2 - 2rc^2)^2}{4} - (b^2 - 2rc^2)\,z - \frac{(b^2 - 2rc^2)^2}{2} = z^2 - \frac{(b^2 - 2rc^2)^2}{4} = -c^2r^2$, $\therefore 4z^2 - b^4 + 4b^2c^2r - 4r^2c^4 = -4c^2r^2$ $4c^2r^2(1 - c^2) + 4b^2c^2r = b^4 - 4z^2$, but $1 - c^2 = b^2$ and $\therefore$ $r^2 + r = \frac{b^4 - 4z^2}{4c^2b^2}$. Now since $r = \text{max.}$ we must have $r^2 + r = \text{max.}$ or its equivalent $\frac{b^4 - 4z^2}{4b^2c^2}$ must be $= \text{max.}$ which can only happen when $z = 0$ $\therefore r^2 + r = \frac{b^4}{4c^2b^2}$.

Solving this quadratic we find $r = \sqrt{\frac{b^2c^2 + b^4}{4c^2b^2}} - \frac{1}{2}$ $\sqrt{\frac{b^2(b^2 + c^2)}{4b^2c^2}} - \tfrac{1}{2} = \sqrt{\frac{b^2 + 1 - b^2}{4c^2}} - \tfrac{1}{2} = \frac{1}{2c} - \frac{1}{2} = \frac{1 - c}{2c} = \frac{1 - c}{2c} \times \frac{1 + c}{1 + c} = \frac{1 - c^2}{2c(1 + c)} = \frac{b^2}{2c(1 + c)}$ $\therefore r = \frac{b^2}{2c(1 + c)}$. Now from equation $y^2 = z + \frac{b^2 - 2rc^2}{2}$ where $z = 0$, we find $y^2 = \frac{b^2 - 2rc^2}{2} = \dfrac{b^2 - \frac{2b^2c^2}{2c(1 + c)}}{2} = \frac{b^2}{2(1 + c)}$ as before.

This is the solution of the problem to find in what direction a body must be projected with a given velocity, that its range, on a given plane, may be the greatest possible.

PROB. (51.) TO FIND x WHEN $\frac{x(a-x)}{a^2}$ IS A MAXIMUM.

Since a^2 is a constant given quantity we must have $x(a-x) = ax - x^2 = \text{max.}$ which let $= r$ $\therefore x^2 - ax = -r$ and solving this quadratic we find, $x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - r}$.

It is manifest that r cannot be greater than $\frac{a^2}{4}$ and therefore we must have $r = \frac{a^2}{4}$ when it is a maximum, $\therefore x = \frac{a}{2}$.

The same may easily be solved without impossible roots.

This is the solution of the optical problem to determine the position and magnitude of the least circle of aberration.

PROB. (52.) A REGULAR HEXAGONAL PRISM IS REGULARLY TERMINATED BY A TRIHEDRAL SOLID ANGLE FORMED BY PLANES, EACH PASSING THROUGH TWO ANGLES OF THE PRISM; FIND THE INCLINATION OF THESE PLANES TO THE AXIS OF THE PRISM, IN ORDER THAT, FOR A GIVEN CONTENT, THE TOTAL SURFACE MAY BE THE LEAST POSSIBLE. (Fig. 39.)

Let *ABCabc* be the base of the prism, *PQRS*, one of the faces of the terminating solid angle passing through the angles *P*, *R*.

Let *S* be the vertex of the pyramid. Draw *SO* perpendicular to the upper surface of the prism. Join *OM*, *RP*, *SQ* intersecting each other in *N*. Then it is easy to see that $MN = NO$ and consequently $SO = QM$, and, as the triangles *POR*, *PMR* are equal, so that, whatever be the inclination of *SQ* to *ON*, the part cut off from them is equal to the part

included in the pyramid SPR, and the content of the whole, therefore, remains constant. We have then to determine the angle ONS, or OSN, so that the total surface shall be a minimum. Let AB, the side of the hexagon, $= a$, AP, the height of the prism, $= b$, $OSN = \theta$. Then $ON = MN = \frac{1}{2}a$, and $SN = \frac{1}{2}a$ co.sec. θ, and $QM = \frac{1}{2}a$ cot. θ. The surface $APBQ = \frac{1}{2}a\,(2b - \frac{1}{2}a \text{ cot.}\theta)$. The surface $PQRS = PR \times SN$ $= \frac{3^{\frac{1}{2}}a^2}{2}$ co. sec. θ. Whence the total surface of the solid is $3a\,(2b - \frac{a}{2} \text{ cot.}\,\theta) + \frac{3^{\frac{1}{2}}a^2}{2} \text{ co. sec.}\,\theta = 6ab - \frac{3a^2}{2} \text{ cot.}\,\theta + \frac{3a^2}{2}\, 3^{\frac{1}{2}} \text{ co. sec.}\,\theta = 6ab + \frac{3a^2}{2}\,(3^{\frac{1}{2}} \text{ co. sec.}\,\theta - \text{cot.}\,\theta) = \text{min.}$ and therefore $\sqrt{3}$ co. sec. θ — cot. θ = min. which let $= r$. Also let cot. $\theta = x$ and $\therefore$ co. sec. $\theta = \sqrt{1 + x^2} \therefore \sqrt{3 + 3x^2} - x = r$, and therefore $3 + 3x^2 = x^2 + 2rx + r^2$ and $x^2 - rx = \frac{r^2 - 3}{2}$. Solving this quadratic we find $x = \frac{r}{2} + \sqrt{\frac{3r^2 - 6}{4}}$ and it is now evident that r cannot be taken so small as to make $3r^2$ less than 6, and therefore we must have $3r^2 = 6$ when r = min. $\therefore r = \sqrt{2}$ and $x = \frac{r}{2} = \frac{1}{\sqrt{2}}$ or cot. $\theta = \frac{1}{\sqrt{2}}$ and tan. $\theta = \sqrt{2}$. Hence tan. $SRN = \frac{1}{\sqrt{2}}$, and $SRQ = 2^{\frac{1}{2}}$.

The same solved without impossible roots.

In the equation $x^2 - rx = \frac{r^2 - 3}{2}$ let $x = y + \frac{r}{2} \therefore x^2 - rx = y^2 + ry + \frac{r^2}{4} - ry - \frac{r^2}{2} = y^2 - \frac{r^2}{4} = \frac{r^2 - 3}{2} \therefore 4y^2 + 6 = 3r^2$ and $r^2 = \frac{4y^2 + 6}{3}$ which is evidently a mini-

mum when $y = 0 \therefore r = \sqrt{\frac{6}{3}} = \sqrt{2}$ and $x = \frac{r}{2} = \frac{1}{\sqrt{2}}$ as before.

This is the celebrated problem of the form of the cells of bees. Maraldi was the first who measured the angles of the faces of the terminating solid angle, and he found them to be 109° 28′ and 70° 32′ respectively. It occurred to Reaumur that this might be the form which, for the same solid content, gives the minimum of surface, and he requested Konig to examine the question mathematically. That Geometer confirmed the conjecture; the result of his calculations agreeing with Maraldi's measurements within 2′. Maclaurin and S. Huillier, by different methods, verified the preceding result, excepting that they showed that the difference of 2′ was owing to an error in the calculations of Konig, and not to a mistake on the part of the bees.

PROB. (53.) TO FIND SUCH A VALUE FOR x AS MAY MAKE $\frac{x}{(a+x)(b+x)}$ A MAXIMUM.

It is evident that when $\frac{x}{(a+x)(b+x)} =$ max. we must have $\frac{(a+x)(b+x)}{x} =$ min. and $\therefore \frac{ab + (a+b)x + x^2}{x} =$ min. or $a + b + \frac{ab + x^2}{x} =$ min. and as $a + b$ is a constant given quantity, we must also have $\frac{ab + x^2}{x} =$ min. which let $= r \therefore x^2 - rx = -ab$. Solving this quadratic we find $x = \frac{r}{2} \pm \sqrt{\frac{r^2}{4} - ab}$, and here it is evident that r cannot be taken so small as to make $\frac{r^2}{4}$ less than $\frac{ab}{2}$ and therefore

when $r =$ min. we must have $\frac{r^2}{4} = ab, \therefore r = 2\sqrt{ab}$ and $x = \frac{r}{2} = \sqrt{ab}.$

The same solved without impossible roots.

• In the equation $x^2 - rx = -ab$ let $x = y + \frac{r}{2}$ $\therefore x^2 - rx = y^2 + ry + \frac{r^2}{4} - ry - \frac{r^2}{2} = y^2 - \frac{r^2}{4} = -ab$ $\therefore r^2 = 4y^2 + 4ab$ which is evidently a minimum when $y=0$, $\therefore r = 2\sqrt{ab}$ and $x = \frac{r}{2} = \sqrt{ab}$ as before.

This is the solution of the dynamical problem to find the magnitude of the body which must be interposed between two others, so that the velocity communicated from the one to the other shall be a maximum.

PROB. (54.) THE DIFFERENCE OF TWO NUMBERS BEING GIVEN, TO FIND IN WHAT CASE THE THIRD PROPORTIONAL TO THE LESS AND THE GREATER OF THEM IS A MINIMUM.

Let $a =$ the given difference of the two numbers, $x =$ greater number, and therefore $x - a =$ the lesser number. We now have $x - a : x :: x : \frac{x^2}{x-a} =$ a third proportional required $=$ min. which let $= r, \therefore x^2 - rx = -ra.$ Solving this quadratic we find $x = \frac{r}{2} \pm \sqrt{\frac{r^2 - 4ra}{4}} = \frac{r}{2} \pm \sqrt{\frac{r}{4}(r - 4a)}$, and here it is evident that r cannot be taken so small as to become less than $4a$, and consequently when $r =$ min. we must have $r = 4a, \therefore x = \frac{r}{2} = \frac{4a}{2} = 2a$

$=$ greater number, and the lesser number $= x - a = 2a - a = a$. Hence it appears that the third proportional required is the least possible when the greater number is double the lesser number.

The same solved without impossible roots.

In the equation $x^2 - rx = -ra$ let $x = y + \frac{r}{2}$ and therefore $x^2 - rx = y^2 + ry + \frac{r^2}{4} - ry - \frac{r^2}{4} = y^2 - \frac{r^2}{4} = -ra$, $\therefore r^2 - 4ra = 4y^2$ and $r = 2a + \sqrt{4y^2 + 4a^2}$ which is evidently a minimum when $y = 0$, $\therefore r = 4a$ and $x = \frac{r}{2}$ $\frac{4a}{2} = 2a$ as before.

PROB. (55.) THE CONTENT OF A CONE BEING GIVEN, FIND ITS FORM WHEN ITS SURFACE IS A MINIMUM.

X the altitude, and y the radius of the base.

Let $\frac{pa^3}{3}$ be the given content $= \therefore \frac{py^2x}{3}$.

Then u = surface = convex surface + base.

But convex surface = sector of circle, of which the radius is the slant side, and the arc the circumference of the base of cone, $\therefore u = py\sqrt{x^2 + y^2} + py^2$. But $y^2 = \frac{a^3}{x}$ $\therefore y^2 + x^2 = \frac{a^3 + x^3}{x}$ $\therefore u = pa^{\frac{3}{2}} \left\{\frac{\sqrt{a^3 + x^3} + a^{\frac{3}{2}}}{x}\right\} =$ min. Now as $pa^{\frac{3}{2}}$ is a constant given quantity we must have

$\frac{\sqrt{a^3 + x^3} + a^{\frac{3}{2}}}{x} =$ min. which let $= r$, and $\therefore$

$\frac{\sqrt{a^3 + x^3} + a^{\frac{3}{2}}}{x} = r$, and $\sqrt{a^3 + x^3} = rx - a^{\frac{3}{2}}$; squaring both sides we find $a^3 + x^3 = r^2x^2 - 2rxa^{\frac{3}{2}} + a^3$, and there-

fore $x^2 = r^2x - 2a^{\frac{3}{2}}$ and $x^2 - r^2x = -2ra^{\frac{3}{2}}$. Solving this quadratic we find $x = \frac{r^2}{2} + \sqrt{\frac{r(r^3 - 8a^{\frac{3}{2}})}{4}}$ and here it is evident that r cannot be taken so small as to make r^3 less than $8a^{\frac{3}{2}}$, and $\therefore r^3 = 8a^{\frac{3}{2}}$ and $r = 2a^{\frac{1}{2}}$ $\therefore r^2 = 4a$ and $x = \frac{r^2}{2} = \frac{4a}{2} = 2a$.

The same solved without impossible roots.

In $x^2 - r^2x = -2ra^{\frac{3}{2}}$ let $x = y + \frac{r^2}{2}$ and therefore $x^2 - r^2x = y^2 + r^2y + \frac{r^4}{4} - r^2y - \frac{r^4}{2} = y^2 - \frac{r^4}{4} = -2ra^{\frac{3}{2}}$, $\therefore r^4 = 4y^2 + 8ra^{\frac{3}{2}}$ which is evidently a minimum when $y = 0$, $\therefore r^4 = 8ra^{\frac{3}{2}}$ and $r = 2a^{\frac{1}{2}}$ and $r^2 = 4a$; therefore $x = \frac{r^2}{2} = 2a$ as before.

CHAPTER II.

PROBLEMS OF MAXIMA AND MINIMA IN THE SOLUTION OF WHICH CUBIC EQUATIONS ARE USED.

Before reading this chapter the article on "Reduction of Equations," in the Introductory Chapter, must be studied with great care, for this reduction is effected in almost every problem which follows.

PROB. (1.) WHAT IS THE FRACTION, THE CUBE OF WHICH BEING SUBTRACTED FROM IT, THE REMAINDER IS THE GREATEST POSSIBLE?

Let $x =$ the fraction required, and the greatest remainder $= r$, $\therefore x - x^3 = r$ and $x^3 - x = -r$, $\therefore x^3 - x + r = 0$. In order to solve this problem merely by means of quadratic equations, let one of the negative roots of this cubic equation $= -a$, and it is evident $x + a$ must exactly divide $x^3 - x + r = 0$, and therefore the following process is obtained.

$$
\begin{array}{lll}
x + a \,\big) & x^3 - x + r = 0 & \big(x^2 - ax + a^2 - 1 = 0 \ldots \text{(A.)} \\
 & \underline{x^3 + ax^2} & \\
 & \quad - ax^2 - x & \\
 & \quad \underline{- ax^2 - a^2 x} & \\
 & \qquad (a^2 - 1)\, x + r & \\
 & \qquad (a^2 - 1)\, x + a^3 - a, & \therefore r \text{ must be } = a^3 - a,
\end{array}
$$

and $\therefore a^2 - 1 = \frac{r}{a}$ $\therefore$ by equation (A) we find $x^2 - ax + \frac{r}{a} = 0$, and $x^2 - ax = -\frac{r}{a}$. Solving this quadratic we find $x = \frac{a}{2} \pm \sqrt{\frac{a^3 - 4r}{4a}}$ and here it is evident

that the greatest value of r is when $a^3 = 4r = 4a^3 - 4a$ $\therefore a = \frac{2}{\sqrt{3}}$ and $x = \frac{a}{2} = \frac{1}{\sqrt{3}}$ = the required value of x.

The same solved without impossible roots.

In the equation $x^2 - ax = -\frac{r}{a}$ let $x = y + \frac{a}{2} \therefore x^2 - ax = y^2 + ax + \frac{a^2}{4} - ay - \frac{a^2}{2} = y^2 - \frac{a^2}{4} = -\frac{r}{a}$ $\therefore r = \frac{a^3}{4} - ay^2$, which is evidently a max. when $y = 0$, $\therefore$ $r = \frac{a^3}{4}$; but $r = a^3 - a$, $\therefore 4a^3 - 4a = a^3$, $\therefore 3a^3 = 4a$ and $a = \frac{2}{\sqrt{3}} \therefore x = \frac{a}{2} = \frac{1}{\sqrt{3}}$ as before.

PROB. (2.) WHAT IS THE FRACTION THE CUBE OF WHICH BEING SUBTRACTED FROM ITS SQUARE, THE REMAINDER IS THE GREATEST POSSIBLE?

Let x = the fraction required, and the greatest remainder $= r \therefore x^2 - x^3 = r \therefore x^3 - x^2 = -r$, or $x^3 - x^2 + r = 0$. In order to eliminate the second term of this equation, let $x = y + \frac{1}{3}$; and by this substitution we find,

$$x^3 = (y + \tfrac{1}{3})^3 = y^3 + y^2 + \tfrac{1}{3}y + \frac{1}{27}$$

$$-x^2 = -(y + \tfrac{1}{3})^2 = -y^2 - \frac{2}{3}y - \frac{1}{9}$$

$$r = \qquad\qquad + r.$$

$\therefore x^3 - x^2 + r = y^3 - \frac{1}{3}y + r - \frac{2}{27} = 0$. Let one of the negative roots of this equation $= -a$, $\therefore$

$$y+a \,\big)\, y^3-\tfrac{1}{3}y+r-\frac{2}{27}=0 \,\big(\, y^2-ay+a^2-\tfrac{1}{3}=0\ldots(\text{A.})$$

$$\begin{array}{l}
\underline{y^3+ay^2} \\
\quad -ay^2-\tfrac{1}{3}y \\
\quad \underline{-ay^2-a^2y} \\
\qquad\qquad (a^2-\tfrac{1}{3})\,y+r-\dfrac{2}{27} \\
\qquad\qquad (a^2-\tfrac{1}{3})\,y+a^3-\dfrac{a}{3} \\
\qquad\qquad \overline{\qquad\qquad\qquad\qquad} \;\therefore\; a^3-\dfrac{a}{3}=r-\dfrac{2}{27}=r'
\end{array}$$

which must also be greatest.

From this equation $a^2-\frac{1}{3}=\dfrac{r'}{a}$ and therefore from equation (A) we find $y^2-ay+\dfrac{r'}{a}=0$, $\therefore\ y^2-ay=-\dfrac{r'}{a}$. Solving this quadratic we find $y=\dfrac{a}{2}\pm\sqrt{\dfrac{a^3-4r'}{4a}}$, and here it is evident that when $r'=$ greatest quantity possible, we must have $a^3=4r'=4a^3-\dfrac{4a}{3}\ \therefore\ 3a^2=\dfrac{4}{3}\ \therefore\ a=\dfrac{2}{3}$.

This problem may be solved without eliminating the second term of the cubic equation, in the following manner.

Let one of the negative roots of the equation $x^3-x^2+r=0=-a$, and therefore—

$$x+a \,\big)\, x^3-x^2+r=0 \,\big(\, x^2-(a+1)\,x+a^2+a=0\ldots(\text{A.})$$

$$\begin{array}{l}
\underline{x^3+ax^2} \\
\quad -(a+1)\,x^2+r \\
\quad \underline{-(a+1)\,x^2-a\,(a+1)\,x} \\
\qquad\qquad\qquad (a^2+a)\,x+r \\
\qquad\qquad\qquad (a^2+a)\,x+a\,(a^2+a) \\
\qquad\qquad\qquad \overline{\qquad\qquad\qquad\qquad} \;\therefore\; a\,(a^2+a)
\end{array}$$

$=a^3+a^2=r$, and $a^2+a=\dfrac{r}{a}$ and therefore from equation (A) we find $x^2-(a+1)\,x+\dfrac{r}{a}=0$ or $x^2-(a+1)\,x=-\dfrac{r}{a}$. Solving this equation we find $x=\dfrac{a+1}{2}\pm$

$\sqrt{\dfrac{a(a+1)^2 - 4r}{4a}}$, and here it is evident that when $r =$ greatest quantity possible, we must have $a\,(a+1)^2 = 4r = 4a^3 + 4a^2 \therefore a^2 + 2a + 1 = 4a^2 + 4a$, or $a = \frac{1}{3}$ and $x = \dfrac{a+1}{2} \pm \sqrt{\dfrac{a(a+1)^2 - 4r}{4a}} = \dfrac{\frac{1}{3}+1}{2} \pm 0 = \frac{2}{3}$ as before.

The same solved without impossible roots.

In the equation $y^2 - ay = -\dfrac{r'}{a}$ let $y = z + \dfrac{a}{2}$ and therefore $y^2 - ay = z^2 + az + \dfrac{a^2}{4} - az - \dfrac{a^2}{2} = z^2 - \dfrac{a^2}{4} = -\dfrac{r'}{a} \therefore r' = \dfrac{a^3}{4} - az^2$ which is evidently a max. when $z = 0, \therefore r' = \dfrac{a^3}{4}$, but $r' = a^3 - \dfrac{a}{8}, \therefore \dfrac{a^3}{4} = a^3 - \dfrac{a}{8}$ $\therefore 3a^3 = \dfrac{4a}{8}$ and $a = \frac{2}{3}$. Now $y = \dfrac{a}{2} = \frac{1}{3}$ and $x = y + \frac{1}{3}$ $= \frac{2}{3}$ as before.

PROB. (3.) TO DETERMINE THE DIMENSIONS OF THE LEAST ISOSCELES TRIANGLE *ACD* THAT CAN CIRCUMSCRIBE A GIVEN CIRCLE. (Fig. 40.)

Let $OS =$ the radius of the given circle $= a$, and $DO =$ the distance of the vertex of the triangle from the centre$=x$. Now the triangles DBC and DOS having the angle ODS common and the angles at B and S right angles, are similar $\therefore DS : OS :: DB : BC$ or $\sqrt{x^2 - a^2} : a :: a+x : BC \therefore BC = \dfrac{a(a+x)}{\sqrt{x^2-a^2}}$ and the area of the triangle $= BC \times DB = \dfrac{a(a+x)^2}{\sqrt{x^2-a^2}}$ which being a min. its square must also be a min., and consequently, $\dfrac{(a+x)^4}{x^2-a^2}$ or its equivalent $\dfrac{(a+x)^3}{x-a}$

is a min. Also let $y = x + a \therefore y - 2a = x - a \therefore \frac{(a+x)^3}{x-a}$ $= \frac{y^3}{y-2a}$ which let $= r \therefore y^3 - ry + 2ar = 0$. Let a negative root of this equation $= -b \therefore y + b$ must exactly divide $y^3 - ry + 2ar = 0 \therefore$ we shall have the following process—

$$
\begin{array}{l}
y + b\,)\; y^3 - ry + 2ar = 0 \;(\, y^2 - by + b^2 - r = 0 \ldots (A.) \\
\qquad \underline{y^3 + by^2} \\
\qquad\quad - by^2 - ry \\
\qquad\quad \underline{- by^2 - b^2y} \\
\qquad\qquad (b^2 - r)\, y + 2ar \\
\qquad\qquad \underline{(b^2 - r)\, y + b(b^2 - r)} \quad \therefore b^3 - br = 2ar
\end{array}
$$

$\therefore r = \frac{b^3}{2a + b}$. Also from equation (A) we have $y^2 - by =$ $r - b^2$, and $\therefore y = \frac{b}{2} \pm \sqrt{r - \frac{3b^2}{4}}$. Now if r be the least possible, we must have $r = \frac{3b^2}{4}$ or $\frac{b^3}{2a+b} = \frac{3b^2}{4}$ or $4b =$ $6a + 3b$ or $b = 6a$, and $y = \frac{b}{2} = \frac{6a}{2} = 3a$, and $x = y - a$ $= 3a - a = 2a =$ the value required.

The same solved without impossible roots.

In the equation $y^2 - by = r - b^2$ let $y = z + \frac{b}{2} \therefore y^2$ $- by = z^2 + bz + \frac{b^2}{4} - bz - \frac{b^2}{2} = z^2 - \frac{b^2}{4} = r - b^2 \therefore$ $r = z^2 + b^2 - \frac{b^2}{4} = z^2 + \frac{3b^2}{4}$, which is evidently a min. when $z = 0$, $\therefore r = \frac{3b^2}{4}$; but $r = \frac{b^3}{2a+b} \therefore \frac{3b^2}{4} = \frac{b^3}{2a+b}$ and $6a + 3b = 4b$, $\therefore b = 6a$ and $y = \frac{b}{2} = \frac{6a}{2} = 3a$, and we therefore find $x = y - a = 3a - a = 2a$ as before.

PROB. (4.) TO DETERMINE THE GREATEST CYLINDER dg THAT CAN BE INSCRIBED IN A GIVEN CONE ADB. (Fig. 41.)

Let $a = BC$, the altitude of the cone

$b = AD$, x the diameter of the cylinder, considered as variable; $p = \left(\frac{3.14159 \text{ \&c.}}{4}\right)$. Now it is evident that the area of the circle $frgs = px^2$, and by similar triangles $AC : BC :: Ad : df$ or $\frac{b}{2} : a :: \frac{b - x}{2} : df = \frac{ab - ax}{b}$ ...... (A.)

And the solid content of the cylinder $= \frac{pabx^2 - pax^3}{b} = \frac{pa}{b} \times (bx^2 - x^3)$ which is a max. $\therefore bx^2 - x^3$ is a max. Let $bx^2 - x^3 = r$, $\therefore x^3 - bx^2 + r = 0$, and $x = y + \frac{b}{3}$, and making this substitution we shall find $x^3 - bx^2 + r = y^3 - \frac{b^2}{3} y + r - \frac{2b^3}{27}$, also let $r - \frac{b^3}{27}$ which is a max. $= r'$ and $\therefore y^3 - \frac{b^2}{3} y + r' = 0$; and proceeding as in prob. (2) this problem may easily be solved. We however subjoin the process.

Let a negative root of this equation $= -c$, $\therefore y + c$ must exactly divide $y^3 - \frac{b^2}{3} y + r' = 0$.

$$y + c \,)\, y^3 - \frac{b^2}{3} y + r' = 0 \,(\, y^2 - cy + c^2 - \frac{b^2}{3} = 0 \ldots (\text{B.})$$

$$\begin{array}{l}
\underline{y^3 + cy^2} \\
- cy^2 - \frac{b^2}{3} y \\
\underline{- cy^2 - c^2 y} \\
\left(c^2 - \frac{b^2}{3}\right) y + r' \\
\underline{\left(c^2 - \frac{b^2}{3}\right) y + c^3 - \frac{b^2 c}{3}}
\end{array} \qquad \therefore r' = c^3 - \frac{b^2 c}{3}$$

and $c^2 - \frac{b^2}{3} = \frac{r'}{c}$ $\therefore$ from equation (B) we find $y^2 - cy + \frac{r'}{c} = 0$, or $y^2 - cy = -\frac{r'}{c}$ $\therefore$ $y = \frac{c}{2} \pm \sqrt{\frac{c^3 - 4r'}{4c}}$. Now in order that r' may be the greatest possible, we must have $4r' = c^3$; but $r' = r - \frac{b^3}{27} = c^3 - \frac{b^2c}{3}$ $\therefore$ $c^3 = 4c^3 - \frac{4b^2c}{3}$ or $c^2 = 4c^2 - \frac{4b^2}{3}$ or $3c^2 = \frac{4b^2}{3}$, $\therefore$ $c = \frac{2b}{3}$, $\therefore$ $y = \frac{c}{2} = \frac{b}{3}$ and $x = y + \frac{b}{3} = \frac{2b}{3}$. Also from equation (A) we have $df = \frac{ab - \frac{2ab}{3}}{b} = a - \frac{2a}{3} = \frac{a}{3}$, and hence it appears that the inscribed cylinder will be the greatest possible when the altitude thereof is just $\frac{1}{3}$ of the altitude of the cone.

The same solved without impossible roots.

In the equation $y^2 - cy = -\frac{r'}{c}$ let $y = z + \frac{c}{2}$ and therefore $y^2 - cy = z^2 + cz + \frac{c^2}{4} - cz - \frac{c^2}{2} = z^2 - \frac{c^2}{4} = -\frac{r'}{c}$ and therefore $r' = \frac{c^3}{4} - cz^2$ which is evidently a maximum when $z = 0$, $\therefore$ $r' = \frac{c^3}{4}$; but $r' = c^3 - \frac{b^2c}{3}$ and $\therefore$ $c^3 = 4c^3 - \frac{4b^2c}{3}$ $\therefore$ $3c^2 = \frac{4b^2}{3}$ and $c = \frac{2b}{3}$. Now $y = \frac{c}{2} = \frac{b}{3}$ and therefore $x = y + \frac{b}{3} = \frac{b}{3} + \frac{b}{3} = \frac{2b}{3}$ as before.

PROB. (5.) TO DETERMINE THE DIMENSIONS OF A CYLINDRIC MEASURE $ABCD$ OPEN AT THE TOP, WHICH SHALL CONTAIN A GIVEN QUANTITY (OF LIQUOR, GRAIN, &c.) UNDER THE LEAST INTERNAL SUPERFICIES POSSIBLE. (Fig. 42.)

Let the diameter $AB = x$, $AD = y$, $p = 3.14159$ &c. and $c =$ the given content of the cylinder. In this case it is evident that px will be the circumference of the base, and consequently, by multiplying it by y, the altitude, we shall find $pxy =$ the concave superficies of the cylinder. It is also evident that since $\frac{px}{2} =$ half the circumference and $\frac{x}{2}$ $=$ half the diameter of the base, we shall have $\frac{px^2}{4} =$ the area of the base, which, being multiplied into the altitude y, we shall have $\frac{px^2y}{4} =$ solid content of the cylinder $= c$, $\therefore y = \frac{4c}{px^2} \therefore pxy = \frac{4c}{x}$ and consequently the whole surface of the cylinder $= \frac{4c}{x} + \frac{px^2}{4}$ which is a minimum. Let $\frac{4c}{x} + \frac{px^2}{4}$ $= r$, and $\therefore 16c + px^3 = 4rx \therefore x^3 - \frac{4r}{p}x + \frac{16c}{p} = 0$. Let one of the negative roots of this equation $= -a$ and therefore $x + a$ must exactly divide $x^3 - \frac{4r}{p}x + \frac{16c}{p} = 0$.

$$x + a \,\big)\, x^3 - \frac{4r}{p}x + \frac{16c}{p} = 0 \,\big(\, x^2 - ax + a^2 - \frac{4r}{p} = 0 \ldots \text{(A.)}$$

$$\begin{array}{l}
\underline{x^3 + ax^2} \\
\quad - ax^2 - \frac{4r}{p}x \\
\quad \underline{- ax^2 - a^2x} \\
\quad \left(a^2 - \frac{4r}{p}\right)x + \\
\quad \underline{\left(a^2 - \frac{4r}{p}\right)x + a^3 - \frac{4ar}{p}}.
\end{array}$$

We therefore

find $a^3 - \frac{4ar}{p} = \frac{16c}{p}$, and $\therefore\ r = \frac{pa^3 - 16c}{4a}$. From equation (A) we find $x^2 - ax = \frac{4r}{p} - a^2$, and $x = \frac{a}{2} \pm \sqrt{\frac{4r}{p} - \frac{3a^2}{4}}$. Now in order that r may be the least possible we must have $\frac{4r}{p} = \frac{3a^2}{4}$ or $\frac{pa^3 - 16c}{ap} = \frac{3a^2}{4}$ or $pa^3 = 64c$ and $a = 4 \times \sqrt[3]{\frac{c}{p}}$ and $x = \frac{a}{2} = 2 \times \sqrt[3]{\frac{c}{p}}$. Now because $px^3 = 8c$ and $px^2y = 4c\ \therefore px^3 = 2px^2y\ \therefore\ x = 2y$ and $y = \sqrt[3]{\frac{c}{p}}$, hence y is known, and from this it appears that the diameter of the base must be just double of the altitude.

The same solved without impossible roots.

In the equation $x^2 - ax = \frac{4r}{p} - a^2$ let $x = y + \frac{a}{2}$ and therefore $x^2 - ax = y^2 + ay + \frac{a^2}{4} - ay - \frac{a^2}{2} = y^2 - \frac{a^2}{4} = \frac{4r}{p} - a^2$, $\therefore\ r = \frac{4py^2 + 3pa^2}{16}$, which is evidently a minimum when $y = 0$, $\therefore\ r = \frac{3pa^2}{16}$; but $r = \frac{pa^3 - 16c}{4a}$, therefore $\frac{3pa^2}{16} = \frac{pa^3 - 16c}{4a}$ or $pa^3 = 64c$ and $a = 4 \times \sqrt[3]{\frac{c}{p}}$ and $x = \frac{a}{2} = 2 \times \sqrt[3]{\frac{c}{p}}$ as before.

PROB. (6.) TO FIND THE LEAST PARABOLA WHICH SHALL CIRCUMSCRIBE A GIVEN CIRCLE. (Fig. 43.)

Since the parabola and the circle touch at P $\therefore$ CP is a normal to the parabola, and Cm is the subnormal $= \frac{1}{2}$ latus rectum. Let $Cm = z$ $\therefore$ equation to the parabola is

$$y^2 = 2z.x \ldots\ldots\ldots\ldots (\text{A.})$$

$$Pm^2 = r^2 - z^2 \therefore Am = \frac{r^2 - z^2}{2z}.$$

$$AD = Am + mC + CD = \frac{r^2 - z^2}{2z} + z + r = \frac{(r+z)^2}{2z}.$$

Now the area of the parabola $EAF = \frac{4}{3} AD.DE$ and $DE = \sqrt{2z.AD}$ $\therefore$ area $EAF = \frac{4}{3} AD \times \sqrt{2z.AD} = \frac{\frac{2}{3}.(r+z)^3}{z}$ $\therefore u = \frac{(r+z)^3}{z} =$ minimum. Let $r + z = y$ $\therefore z = y - r$ $\therefore \frac{y^3}{y-r} =$ minimum $= u$, and $\therefore y^3 - uy + ur = 0$. Let one of the negative roots of this equation $= -a$, and therefore $y + a$ must exactly divide the equation $y^3 - uy + ru = 0$.

$$y + a \,)\, y^3 - uy + ru = 0 \,(\, y^2 - ay + a^2 - u = 0 \ldots (\text{B.})$$

$$\begin{array}{l} \underline{y^3 + ay^2} \\ \quad - ay^2 - uy \\ \quad \underline{- ay^2 - a^2y} \\ \qquad (a^2 - u)\, y + ru \\ \qquad \underline{(a^2 - u)\, y + a^3 - au} \end{array}$$

and therefore we must have $a^3 - au = ru$ $\therefore u = \frac{a^3}{a+r}$.

Now solving this quadratic (B) we find $y = \frac{a}{2} \pm \sqrt{u - \frac{3a^2}{4}}$; and in order that u may become a minimum, we must have $u = \frac{3a^2}{4}$ $\therefore \frac{a^3}{a+r} = \frac{3a^2}{4}$ or $\frac{a}{a+r} = \frac{3}{4}$ $\therefore$

$3a + 3r = 4a \therefore a = 3r \therefore y = \frac{a}{2} = \frac{3r}{2} \therefore z = y - r = \frac{3r}{2} - r = \frac{r}{2}$ .... Q. E. D.

The same solved without impossible roots.

In the equation $y^2 - ay + a^2 - u = 0$ or $y^2 - ay = u - a^2$ let $y = w + \frac{a}{2}$, and therefore $y^2 - ay = w^2 + aw + \frac{a^2}{4} - aw - \frac{a^2}{2} = w^2 - \frac{a^2}{4} = u - a^2 \therefore u = w^2 + \frac{3a^2}{4}$, which is evidently a minimum when $w = 0$, $\therefore u = \frac{3a^2}{4}$; but $u = \frac{a^3}{a+r} \therefore \frac{a^3}{a+r} = \frac{3a^2}{4}$ or $4a = 3a + 3r$ and $a = 3r$, and therefore $y = \frac{a}{2} = \frac{3r}{2}$ and $z = y - r = \frac{r}{2}$ as before.

PROB. (7.) THE FOUR EDGES OF A RECTANGULAR PIECE OF LEAD, a INCHES IN LENGTH AND b INCHES IN BREADTH, ARE TO BE TURNED UP PERPENDICULARLY SO AS TO FORM A VESSEL THAT SHALL HOLD THE GREATEST QUANTITY OF WATER, HOW MUCH OF THE EDGE MUST BE TURNED UP?

It must be observed that the piece of lead is a rectangular sheet, and consequently when x = breadth of edge turned up: then $x(a - 2x)\ (b - 2x)$ = content of vessel = maximum $\therefore 4x^3 - 2(a + b)\,x^2 + abx = 4r$ = maximum, or $x^3 - \frac{a+b}{2}x^2 + \frac{ab}{4}x - r = 0$. Let $x = y + \frac{a+b}{6}$

$$\therefore x^3 = y^3 + \frac{a+b}{2}y^2 + \frac{(a+b)^2}{12}y + \frac{(a+b)^3}{216}$$

$$-\frac{a+b}{2}x^2 = -\frac{a+b}{2}y^2 - \frac{(a+b)^2}{6}y - \frac{(a+b)^3}{72}$$

$$+\frac{ab}{4}x = \dots\dots\dots\dots\dots + \frac{ab}{4}y + \frac{ab(a+b)}{24}$$
$$-r = \qquad - r.$$

$$\therefore y^3 - \frac{(a+b)^2-3ab}{12}y = r + \frac{(a+b)^3}{72} - \frac{(a+b)^3}{216} - \frac{ab(a+b)}{24}.$$

Now r is a maximum; and besides r the remaining terms of the second member of the equation are constant and given quantities, and consequently the whole of the second member must be a maximum when r is so, and therefore when we suppose r' = the whole second member, we must have r' = maximum.

Let $n = \frac{(a+b)^2-3ab}{12} = \frac{a^2-ab+b^2}{12}$ $\therefore y^3 - ny - r' = 0$. Suppose that one of the positive roots of this equation is $= c$, and therefore $y - c$ must exactly divide $y^3 - ny - r' = 0$.

$$y - c \,)\, y^3 - ny - r' = 0 \,(\, y^2 + cy + c^2 - n = 0 \dots \text{(A.)}$$
$$\begin{array}{l} y^3 - cy^2 \\ \hline cy^2 - ny \\ cy^2 - c^2y \\ \hline (c^2-n)\,y - r' \\ (c^2-n)\,y - (c^3 - cn) \\ \hline \end{array}$$
$\therefore c^3 - cn = r'$ and

$\therefore \frac{r'}{c} = c^2 - n$ and consequently from equation (A) we find $y^2 + cy + \frac{r'}{c} = 0$ and $\therefore y = -\frac{c}{2} \pm \sqrt{\frac{c^3-4r'}{4c}}$.

Now it is evident that when r' or $4r'$ is maximum, we must have $c^3 = 4r'$ $\therefore c^3 = 4c^3 - 4cn$ or $c = \pm 2\sqrt{\frac{n}{3}}$ $\therefore y = -\sqrt{\frac{n}{3}} = -\frac{\sqrt{a^2-ab+b^2}}{6}$ and $x = y + \frac{a+b}{6} = \frac{1}{6}\left\{a + b - \sqrt{a^2 - ab + b^2}\right\}$. We have here taken the nega-

tive value of y, because on this supposition only can the equation $y^3 - ny = -r'$ be a maximum.

The same solved without impossible roots.

In the equation $y^2 + cy + \frac{r'}{c} = 0$ or $y^2 + cy = \frac{r'}{c}$ let $y = z - \frac{c}{2}$ $\therefore$ $y^2 + cy = z^2 - cz + \frac{c^2}{4} + cz - \frac{c^2}{2} = z^2 - \frac{c^2}{4} = -\frac{r'}{c}$ $\therefore$ $\frac{r'}{c} = \frac{c^2}{4} - z^2$ which is evidently a max. when $z = 0$, $\therefore$ $\frac{r'}{c} = \frac{c^2}{4}$; but $\frac{r'}{c} = \frac{c^3 - cn}{c} = \frac{c^2}{4}$ or $c^2 - n = \frac{c^2}{4}$ or $3c^2 = 4n$ $\therefore$ $c = \pm 2\sqrt{\frac{n}{3}}$ and $y = -\frac{c}{2} = -\sqrt{\frac{n}{3}} = -\frac{\sqrt{a^2 - ab + b^2}}{6}$ and $x = y + \frac{a+b}{6} = \frac{1}{6}\left\{a + b - \sqrt{a^2 - ab + b^2}\right\}$ as before.

PROB. (8.) TO INSCRIBE THE GREATEST RECTANGLE IN A GIVEN PARABOLA *BPAqD*. (Fig. 44.)

Let $Am = x$ $\therefore$ $Pm = 2\sqrt{mx}$ and $Pq = 4\sqrt{mx}$ $\therefore$ $Pn = mc = Ac - Am = b - x$ $\therefore$ area nq of the required rectangle $= 4(b - x)\sqrt{mx} = \text{max.}$ $\therefore$ $(b - x)\sqrt{x} = \text{max}$ $\therefore$ $(b - x)^2 x = b^2x - 2bx^2 + x^3 = r = \text{max.}$ Let $a =$ one of the positive roots of this equation, $\therefore$ $x - a$ must exactly divide $b^2x - 2bx^2 + x^3 - r = 0$

$$x-a \,)\; x^3 - 2bx^2 + b^2x - r = 0 \;(\; x^2 + (a-2b)x + (a-b)^2 = 0 \ldots (\text{A.})$$

$$\begin{array}{l}
\underline{x^3 - ax^2} \\
\quad (a - 2b)\,x^2 + b^2x \\
\quad \underline{(a - 2b)\,x^2 - a(a - 2b)\,x} \\
\qquad\qquad (a - b)^2 x - r \\
\qquad\qquad \underline{(a - b)^2 x - a\,(a - b)^2} \quad \therefore\; r = a\,(a - b)^2
\end{array}$$

and $\frac{r}{a} = (a-b)^2$ $\therefore$ from equation (A) we find $x^2 + (a-2b)$ $x + \frac{r}{a} = 0$. Solving this quadratic we find $x = -\frac{(a-2b)}{2}$ $\pm\sqrt{\frac{a(a-2b)^2 - 4r}{4a}}$ and here it is evident that when $r =$ max. then $a(a-2b)^2 = 4r = 4a(a-b)^2$ $\therefore$ $a - 2b = \pm$ $2(a-b)$.

1st. $a - 2b = 2a - 2b$ $\therefore$ $a = 0$ and $x = b$.

2nd. $a - 2b = 2b - 2a$ $\therefore$ $a = \frac{4b}{3}$ and $x = \frac{b}{3}$.

By a reference to the annexed diagram, it is evident that $x = \frac{b}{3}$ corresponds to max. and $x = b$ to min.

The same solved without impossible roots.

In $x^2 + (a-2b)\,x + \frac{r}{a} = 0$ let $x = y - \frac{(a-2b)}{2}$ and $\therefore$ $x^2 + (a-2b)\,x = y^2 - (a-2b)\,y + \frac{(a-2b)^2}{4} + (a-2b)$ $y - \frac{(a-2b)^2}{2} = y^2 - \frac{(a-2b)^2}{4} = -\frac{r}{a}$ $\therefore$ $r = \frac{a(a-2b)^2}{4}$ $- ay^2$ which is $=$ max. when $y = 0$, $\therefore$ $r = \frac{a(a-2b)^2}{4}$; but $r = a(a-b)^2$ $\therefore$ $a(a-b)^2 = \frac{a(a-2b)^2}{4}$ or $a - b = \pm$ $\frac{a-2b}{2}$ and

1st. $2a - 2b = a - 2b$ $\therefore$ $a = 0$ and $x = b$.

2nd. $a - 2b = 2b - 2a$ $\therefore$ $a = \frac{4b}{3}$ and $x = \frac{b}{3}$ as before.

PROB. (9.) TO DIVIDE A GIVEN LINE INTO TWO SUCH PARTS THAT THEIR PRODUCT MULTIPLIED INTO THE DIFFERENCE OF THEIR SQUARES SHALL BE A MAXIMUM.

Let $2a$ be the given liue and $a + x$ and $a - x$ the required parts. Now by the problem $(a^2 - x^2) \times 4ax = \text{max.}$ $\therefore x(a^2 - x^2) = \text{max.}$ which let $= r$ or $x^3 - a^2x + r = 0$. Also let $b =$ one of the negative roots of this equation.

$$x + b \,\big)\, x^3 - a^2x + r = 0 \,\big(\, x^2 - bx + b^2 - a^2 \text{.........(A.)}$$

$$\begin{array}{l} x^3 + bx^2 \\ \hline \quad - bx^2 - a^2x \\ \quad - bx^2 - b^2x \\ \hline \qquad (b^2 - a^2)\, x + r \\ \qquad (b^2 - a^2)\, x + b(b^2 - a^2) \\ \hline \end{array} \quad \therefore r = b(b^2 - a^2)$$

$\therefore b^2 - a^2 = \dfrac{r}{b}$ and from equation (A) $x^2 - bx + \dfrac{r}{b} = 0$ or $x^2 - bx = -\dfrac{r}{b}$ and $x = \dfrac{b}{2} \pm \sqrt{\dfrac{b^3 - 4r}{4b}}$. Now it is evident that when $r = \text{max.}$ we must have $b^3 = 4r = 4b(b^2 - a^2)$ $\therefore b^2 = 4b^2 - 4a^2$ $\therefore 4a^2 = 3b^2$ $\therefore b = \dfrac{2a}{\sqrt{3}}$ and $x = \dfrac{b}{2} = \dfrac{a}{\sqrt{3}}$.

The same solved without impossible roots.

In the equation $x^2 - bx = -\dfrac{r}{b}$ let $x = y + \dfrac{b}{2}$ and therefore we find $x^2 - bx = y^2 + by + \dfrac{b^2}{4} - by - \dfrac{b^2}{2} = y^2 - \dfrac{b^2}{4} = -\dfrac{r}{b}$, and $\therefore r = \dfrac{b^3}{4} - by^2 = \text{max.}$ when $y = 0$, $\therefore r = \dfrac{b^3}{4}$; but $r = b(b^2 - a^2)$ $\therefore \dfrac{b^3}{4} = b(b^2 - a^2)$ or $b^2 =$

$4(b^2 - a^2)$ $\therefore$ $3b^2 = 4a^2$ and $b = \dfrac{2a}{\sqrt{3}}$ $\therefore$ $x = \dfrac{b}{2} = \dfrac{a}{\sqrt{3}}$ as before.

PROB. (10.) TO INSCRIBE THE GREATEST ELLIPSE IN A GIVEN ISOSCELES TRIANGLE. (Fig. 45.)

Let $Da = 2x$, $cb = y$, $AD = a$, $DB = b$. Now by the property of the Ellipse we have $cn = \dfrac{ca^2}{cA} = \dfrac{x^2}{a-x}$ $\therefore$ $an = \dfrac{ax - 2x^2}{a - x}$ $Dn = \dfrac{ax}{a-x}$. But $\dfrac{BD^2}{AD^2} \times An^2 = Pn^2 = \dfrac{y^2}{x^2}$ $(an \times nD)$ $\therefore$ $b^2\left(\dfrac{a-2x}{a-x}\right)^2 = y^2\,\dfrac{(a-2x)a}{(a-x)^2}$ $\therefore$ $\pi\, yx = \dfrac{\pi b}{\sqrt{a}}$ $x\sqrt{a-2x} = \text{max.}$ $\therefore$ $x^2(a-2x) = ax^2 - 2x^3 = \text{max.} = 2r$ or $x^3 - \dfrac{a}{2}x^2 + r = 0$. Let b = one of the negative roots of this equation.

$$
\begin{array}{l}
x + b\,\big)\; x^3 - \dfrac{a}{2}x^2 + r = 0 \;\big(\, x^2 - \left(b + \dfrac{a}{2}\right)x + b\left(b + \dfrac{a}{2}\right) = 0, \text{(A.)} \\
\qquad\quad \underline{x^3 + bx^2} \\
\qquad\qquad - \left(b + \dfrac{a}{2}\right)x^2 + r \\
\qquad\qquad \underline{- \left(b + \dfrac{a}{2}\right)x^2 - b\left(b + \dfrac{a}{2}\right)x} \\
\qquad\qquad\qquad\qquad b\left(b + \dfrac{a}{2}\right)x + r \\
\qquad\qquad\qquad\qquad \underline{b\left(b + \dfrac{a}{2}\right)x + b^2\left(b + \dfrac{a}{2}\right)}
\end{array}
$$

$\therefore$ $r = b\left(b^2 + \dfrac{ab}{2}\right)$ $\therefore$ $\dfrac{r}{b} = b^2 + \dfrac{ab}{2}$ and hence from equation (A) we find $x^2 - \left(b + \dfrac{a}{2}\right)x = -\dfrac{r}{b}$ and $\therefore$ $x = \dfrac{2b + a}{4}$

$+ \frac{1}{2}\sqrt{\left(b + \frac{a}{2}\right)^2 - \frac{4r}{b}}$ when $r =$ max. then $b\left(b + \frac{a}{2}\right)^2 = 4r$ $= 4b^2\left(b + \frac{a}{2}\right)$ $\therefore b + \frac{a}{2} = 4b \therefore b = \frac{a}{6}$ and $2b = \frac{a}{3} \therefore$

$$x = \frac{2b + a}{4} = \frac{\frac{a}{3} + a}{4} = \frac{\frac{4a}{3}}{4} = \frac{4a}{12} = \frac{a}{3} \therefore x = \frac{a}{3}.$$

The same solved without impossible roots.

In the equation $x^2 - \left(b + \frac{a}{2}\right)x = -\frac{r}{b}$ let $x = y + \frac{b + \frac{a}{2}}{2}$ $\therefore x^2 - \left(b + \frac{a}{2}\right)x = y^2 + \left(b + \frac{a}{2}\right)y + \frac{\left(b + \frac{a}{2}\right)^2}{4}$ $- \left(b + \frac{a}{2}\right)y - \frac{\left(b + \frac{a}{2}\right)^2}{2} = y^2 - \frac{\left(b + \frac{a}{2}\right)^2}{4} = -\frac{r}{b}$ $\therefore r = \frac{b\left(b + \frac{a}{2}\right)^2}{4} - by^2 =$ max. when $y = 0$, $\therefore r = \frac{b\left(b + \frac{a}{2}\right)^2}{4}$; but $r = b^2\left(b + \frac{a}{2}\right) \therefore b^2\left(b + \frac{a}{2}\right) = \frac{b\left(b + \frac{a}{2}\right)^2}{4}$ and $4b = b + \frac{a}{2} \therefore b = \frac{a}{6} \therefore x = \frac{2b + a}{4}$ $= \frac{a}{3}$ as before.

PROB. (11.) WITHIN A GIVEN PARABOLA TO INSCRIBE THE GREATEST PARABOLA, THE VERTEX OF THE LATTER BEING AT THE BISECTION OF THE BASE OF THE FORMER. (Fig. 46.)

Let ABC be the given parabola of which the axis $BD = a$ and $4m$ the latus rectum are known. Let $Br = x$ $\therefore$ $rD = a - x$ and $mr = y = 2\sqrt{mx}$ $\therefore$ the area of the required parabola $mDr = \frac{2y(a-x)}{3} = \frac{4\sqrt{m}}{3}(a-x)\sqrt{x} = \max.$ $\therefore$ $(a-x)\sqrt{x}$ or $(a-x)^2 x = \max.$ Now let $(a-x)^2 x = r$ $\therefore$ $x^3 - 2ax^2 + a^2 x - r = 0$; also let $b =$ one of the positive roots of this equation, and consequently $x - b$ must exactly divide it.

$$x - b \,)\, x^3 - 2ax^2 + a^2x - r = 0 \,(\, x^2 + (b-2a)x + (a-b)^2 = 0, \text{(A.)}$$
$$\begin{array}{l} x^3 - bx^2 \\ \hline (b-2a)x^2 + a^2x \\ (b-2a)x^2 - b(b-2a)x \\ \hline (a-b)^2x - r \\ (a-b)^2x - b(a-b)^2 \\ \hline \end{array} \quad \therefore r = b(a-b)^2$$

or $(a-b)^2 = \frac{r}{b}$. Now from equation (A) we find $x^2 + (b-2a)x = -\frac{r}{b}$ and $x = -\frac{b-2a}{2} + \sqrt{\frac{b(b-2a)^2 - 4r}{4b}}$, and in order that r may be a max. we must have $b(b-2a)^2 = 4r = 4b(a-b)^2$ or $b - 2a = 2a - 2b$, or $b = \frac{4a}{3}$ $\therefore$

$$x = -\frac{b-2a}{2} = \frac{\frac{2a}{3}}{2} = \frac{a}{3}.$$

The same may be solved without impossible roots.

In the equation $x^2 + (b - 2a)\, x = -\frac{r}{b}$ let $x = y - \frac{b-2a}{2}$ $\therefore x^2 + (b - 2a)\, x = y^2 - (b - 2a)\, y + \frac{(b-2a)^2}{4} + (b - 2a)\, y - \frac{(b-2a)^2}{2} = y^2 - \frac{(b-2a)^2}{4}$ $\therefore r = \frac{b(b-2a)^2}{4} - by^2 =$ max. when $y = 0$, and $\therefore r = \frac{b(b-2a)^2}{4}$; but $r = b(a-b)^2$ $\therefore 4b\,(a-b)^2 = b(b-2a)^2$ or $4a^2 - 8ab + 4b^2 = b^2 - 4ab + 4a^2$ and $4ab = 3b^2$ $\therefore b = \frac{4a}{3}$ and $x = -$

$$\frac{b-2a}{2} = \frac{\frac{2a}{3}}{2} = \frac{a}{3}$$ as before.

PROB. (12.) TO INSCRIBE THE GREATEST CONE WITHIN A GIVEN SPHERE. (Fig. 47.)

Let *ArcB* be the required cone inscribed within the sphere *AmcB*. Let the diameter *Bm* of the given sphere $= 2a$, $BD = x$ $\therefore Dm = 2a - x$, $p = 3.14$ &c. Now by the property of the circle $AD^2 = 2ax - x^2$ $\therefore 4AD^2 = 4(2ax - x^2)$ $\therefore$ the area of the base *Arc* of the required cone $= \frac{p}{4}\, 4(2ax - x^2)$ $= p\,(2ax - x^2)$ $\therefore$ content of the cone $= \frac{p}{3}\, x\,(2ax - x^2)$ $= \frac{p}{3}\,(2ax^2 - x^3) =$ max. $\therefore 2ax^2 - x^3 =$ max. which let $= r$ $\therefore x^3 - 2ax^2 + r = 0$; also let $b =$ one of the negative values of this equation, and consequently $x + b$ must exactly divide it.

$$
\begin{array}{r|l}
x+b & x^3-2ax^2+r=0 \quad \underline{|\, x^2-(b+2a)x+b(b+2a)=0,} \text{ (A.)} \\
& \underline{x^3+bx^2} \\
& \quad -(b+2a)x^2+r \\
& \quad \underline{-(b+2a)x^2-b(b+2a)x} \\
& \qquad\qquad b(b+2a)x+r \\
& \qquad\qquad \underline{b(b+2a)x+b^2(b+2a)} \quad \therefore r=b^2
\end{array}
$$

$(b+2a)$ or $b(b+2a) = \frac{r}{b}$. Now from equation (A) we find $x^2 - (b+2a)x = -\frac{r}{b}$ or $x = \frac{b+2a}{2} \pm \sqrt{\frac{b(b+2a)^2 - 4r}{4b}}$, and in order that r or $4r$ may become a max. we must have $4r = b(b+2a)^2$ or $4b^2(b+2a) = b(b+2a)^2$ $\therefore$ $4b = b + 2a$ or $b = \frac{2a}{3}$ $\therefore$ $x = \frac{b+2a}{2} = \frac{\frac{2a}{3}+2a}{2} = \frac{4a}{3}$.

The same solved without impossible roots.

In the equation $x^2 - (b+2a)x = -\frac{r}{b}$ let $x = y + \frac{b+2a}{2}$ and therefore we find by substitution

$x^2 - (b+2a)x = y^2 + (b+2a)y + \frac{(b+2a)^2}{4} - (b+2a)y - \frac{(b+2a)^2}{2} = y^2 - \frac{(b+2a)^2}{4} = -\frac{r}{b}$, $\therefore$ $r = \frac{b(b+2a)^2}{4} - by^2 =$ max. when $y = 0$ $\therefore$ $r = \frac{b(b+2a)^2}{4}$; but $r = b^2(b+2a)$ and $\therefore$ $b^2(b+2a) = \frac{b(b+2a)^2}{4}$ $\therefore$ $b = \frac{2a}{3}$ and $x = \frac{b+2a}{2} = \frac{4a}{3}$ as before.

PROB. (13.) GIVEN THE SURFACE OF A CYLINDER TO FIND ITS FORM, THAT ITS VOLUME MAY BE A MAXIMUM.

Let the whole surface of the cylinder $= s$ and $x =$ diameter of its base. Now it is evident that the areas of the two opposite circles of the cylinder $= \frac{px^2}{2}$ where $p = 3.14$ &c., the circumference of the base $= px$, and the convex surface $= s - \frac{px^2}{2} = \frac{2s - px^2}{2}$ which divided by px, the circumference of the base, gives the altitude $= \frac{2s - px^2}{2px}$. Now multiplying this value of the altitude into $\frac{px^2}{4}$, the area of the base, we find the content of the cylinder $= \frac{px^2}{4} \times \frac{2s - px^2}{2px}$ $= \frac{2sx - px^3}{8} =$ max. and $\therefore$ $2sx - px^3 =$ max. or $\frac{2s}{p}x - x^3 =$ max. which let $= r$, $\therefore$ $x^3 - \frac{2s}{p}x + r = 0$. Now let $a =$ one of the negative roots of this equation and consequently $x + a$ must exactly divide it.

$$
\begin{array}{l}
x + a \,\big)\; x^3 - \frac{2s}{p}x + r = 0 \;\big(\; x^2 - ax + a^2 - \frac{2s}{p} = 0 \ldots \text{(A.)} \\
\qquad\quad \underline{x^3 + ax^2} \\
\qquad\quad - ax^2 - \frac{2s}{p}x \\
\qquad\quad \underline{- ax^2 - a^2x} \\
\qquad\qquad \left(a^2 - \frac{2s}{p}\right)x + r \\
\qquad\qquad \left(a^2 - \frac{2s}{p}\right)x + a\left(a^2 - \frac{2s}{p}\right) \\
\qquad\qquad \overline{\qquad\qquad\qquad\qquad} \;\therefore\; r = a\left(a^2 - \frac{2s}{p}\right)
\end{array}
$$

or $\frac{r}{a} = a^2 - \frac{2s}{p}$. Now from equation (A) $x^2 - ax = -\frac{r}{a}$ or $x = \frac{a}{2} \pm \sqrt{\frac{a^3 - 4r}{4a}}$, and in order that r may be a max. we must have $a^3 = 4r = 4a\left(a^2 - \frac{2s}{p}\right)$ and $\therefore$ $a = \frac{2\sqrt{2s}}{\sqrt{3p}}$ and $x = \frac{a}{2} = \sqrt{\frac{2s}{3p}}$. Writing this value of x in the equation altitude $= \frac{2s - px^2}{2px}$ we shall find altitude $= \sqrt{\frac{2s}{3p}}$ and hence it appears that altitude $=$ the diameter of the base.

The same solved without impossible roots.

In the equation $x^2 - ax = -\frac{r}{a}$ let $x = y + \frac{a}{2}$ and therefore $x^2 - ax = y^2 + ay + \frac{a^2}{4} - ay - \frac{a^2}{2} = y^2 - \frac{a^2}{4} = -\frac{r}{a}$ $\therefore$ $r = \frac{a^3}{4} - ay^2 =$ max. when $y = 0$, $\therefore$ $r = \frac{a^3}{4}$; but $r = a\left(a^2 - \frac{2s}{p}\right)$ and $\therefore$ $\frac{a^3}{4} = a\left(a^2 - \frac{2s}{p}\right)$ and $a = 2\sqrt{\frac{2s}{3p}}$ $\therefore$ $x = \frac{a}{2} = \sqrt{\frac{2s}{3p}}$ as before.

PROB. (14.) TO PROVE THAT THE ALTITUDE OF THE GREATEST CYLINDER WHICH CAN BE INSCRIBED IN A GIVEN SPHERE, IS EQUAL TO $2r\sqrt{\frac{1}{3}}$; r BEING THE RADIUS. (Fig. 48.)

Let the altitude mn of the cylinder required $= 2x$, and r being the centre of the sphere $rn = x$ $\therefore$ $Bn = \sqrt{r^2 - x^2}$ $=$ the radius of the base of the cylinder, and $\therefore$ the area of

the base $= p\ (r^2 - x^2)$ where $p = 3.14$, &c. Now since altitude of the cylinder $= 2x$, its contents must be $= 2p\ (r^2x - x^3) =$ max. which let $= 2pq$, $\therefore x^3 - r^2x + q = 0$. Let one of the negative values of this equation $= a$, and consequently $x + a$ must exactly divide it.

$$
\begin{array}{l}
x + a \,\big)\, x^3 - r^2x + q = 0 \,\big(\, x^2 - ax + a^2 - r^2 = 0, \ldots \text{(A.)} \\
\qquad\quad \underline{x^3 + ax^2} \\
\qquad\qquad - ax^2 - r^2x \\
\qquad\qquad \underline{- ax^2 - a^2x} \\
\qquad\qquad\qquad (a^2 - r^2)\ x + q \\
\qquad\qquad\qquad \underline{(a^2 - r^2)\ x + a^3 - ar^2} \quad \therefore\ q = a^3 - ar^2
\end{array}
$$

$\therefore \dfrac{q}{a} = a^2 - r^2$. Now from equation (A) we find $x^2 - ax = -\dfrac{q}{a}$ $\therefore x = \dfrac{a}{2} \pm \sqrt{\dfrac{a^3 - 4q}{4a}}$, and in order that q may be the greatest possible, a^3 must be $= 4q = 4a^3 - 4ar^2$ $\therefore$ $a^2 = 4a^2 - 4r^2$, $\therefore a = 2r\sqrt{\dfrac{1}{3}}$ $\therefore x = \dfrac{a}{2} = r\sqrt{\dfrac{1}{3}}$ $\therefore$ $2x =$ altitude required $= 2r\sqrt{\dfrac{1}{3}}$.

The same solved without impossible roots.

In the equation $x^2 - ax = -\dfrac{q}{a}$ let $x = y + \dfrac{a}{2}$ $\therefore x^2 - ax = y^2 + ay + \dfrac{a^2}{4} - ay - \dfrac{a^2}{2} = y^2 - \dfrac{a^2}{4} = -\dfrac{q}{a}$ or $q = \dfrac{a^3}{4} - ay^2 =$ max. when $y = 0$, $\therefore q = \dfrac{a^3}{4}$; but $q = a^3 - ar^2$ and $\therefore \dfrac{a^3}{4} = a^3 - ar^2$ $\therefore a = \dfrac{2r}{\sqrt{3}}$ $\therefore x = \dfrac{a}{2} = \dfrac{r}{\sqrt{3}} = r\sqrt{\dfrac{1}{3}}$ as before.

PROB. (15.) A CANDLE STANDS ON A HORIZONTAL TABLE DIRECTLY OVER A POINT, AT A GIVEN DISTANCE FROM A SMALL OBJECT ON THE TABLE; WHAT OUGHT TO BE THE HEIGHT OF THE FLAME WHEN THE OBJECT IS ILLUMINATED THE MOST POSSIBLE? (Fig. 49.)

Let A be the object on the table, B the point under the candle, and C the flame, considered as condensed at a point. The intensity of the illumination on the object A depends on its distance from C, and on the angle which the rays make with the surface (supposed to be horizontal). By the principles of Optics, the intensity at different distances, the angle of obliquity being the same, will be inversely as the square of the distance; with different degrees of obliquity, the distance being the same, as the sine of the angle which the rays make with the surface. Therefore the intensity, as depending on both obliquity and distance, will be expressed by $\frac{1}{AC^2}$ sin. $CAB = \frac{BC}{AC^3}$. But $a = AB$, $n = \sin. CAB$, then the illuminating power on the surface at $A = \frac{BC}{AC} \times \frac{AB^2}{AC^2} \times \frac{1}{AB^2}$ $= \sin. n \frac{\cos.^2 n}{a^2} = \max.$ $\therefore$ $\sin. n \cos.^2 n = \sin. n\ (1 - \sin.^2 n)$ $= \sin. n - \sin.^3 n = \max. = r$. Now let $\sin. n = x$, $\therefore x - x^3$ $= r$, $\therefore x^3 - x + r = 0$. By problem (1) when $r = \max.$ then $x = \frac{1}{\sqrt{3}}$ $\therefore \sin. n = \frac{1}{\sqrt{3}}$. By the trigonometrical tables $n = 35°\ 16'$; this gives $BC = AB \times \frac{1}{\sqrt{2}} = AB \times$ $\cdot 71$ nearly; so that the height of the flame must be about $\frac{7}{10}$ of the distance AB.

The same may be solved without impossible roots as in problem (1).

PROB. (16.) TO DIVIDE 12 INTO TWO PARTS, SO THAT THE LESSER MULTIPLIED BY THE SQUARE OF THE GREATER SHALL BE A MAXIMUM.

Let x = greater part $\therefore$ $12 - x$ = lesser part. Now it is required to find such a value for x that $(12 - x)\, x^2$ or $12x^2 - x^3$ may be a maximum. Let $12x^2 - x^3 = r$ $\therefore$ $x^3 - 12x^2 + r = 0$. Suppose that a = a negative root of this equation, and consequently $x + a$ must exactly divide it.

$$
\begin{array}{l}
x + a \,)\, x^3 - 12x^2 + r = 0 \,(\, x^2 - (a+12)\,x + a(a+12) = 0, \text{ (A.)} \\
\quad\; \underline{x^3 + ax^2} \\
\qquad - (a + 12)\, x^2 + r \\
\qquad \underline{- (a + 12)\, x^2 - a\,(a + 12)x} \\
\qquad\qquad\qquad a\,(a + 12)\, x + r \\
\qquad\qquad\qquad \underline{a\,(a + 12)\, x + a^2\,(a + 12)}
\end{array}
$$

$\therefore$ $r = a^2\,(a + 12)$ $\therefore$ $\dfrac{r}{a} = a\,(a + 12)$. Now from equation (A) we find $x^2 - (a + 12)\, x = -\dfrac{r}{a}$ or $x = \dfrac{a + 12}{2} \pm \sqrt{\dfrac{a\,(a + 12)^2 - 4r}{4a}}$, and in order that r or $4r$ may be a max. we must have $a\,(a + 12)^2 = 4r = 4a^2\,(a + 12)$ or $a = 4$ and $x = \dfrac{a + 12}{2} = 8$.

The same may be solved without impossible roots.

In the equation $x^2 - (a + 12)\, x = -\dfrac{r}{a}$ let $x = y + \dfrac{a + 12}{2}$, and $\therefore$ $x^2 - (a + 12)\, x = y^2 + (a + 12)\, y + \dfrac{(a + 12)^2}{4} - (a + 12)\, y - \dfrac{(a + 12)^2}{2} = y^2 - \dfrac{(a + 12)^2}{4} = -\dfrac{r}{a}$ $\therefore$ $r = \dfrac{a(a + 12)^2}{4} - ay^2$ = max. when $y = 0$, and $\therefore$ $r = \dfrac{a(a + 12)^2}{4}$;

but $r = a^2(a+12)$ $\therefore$ $\frac{a(a+12)^2}{4} = a^2(a+12)$ $\therefore$ $a = 4$, and $x = \frac{a+12}{2} = 8$ as before.

PROB. (17.) WHAT ARE THE VALUES OF x WHEN $\frac{x^3}{3} - \frac{3x^2}{2} + 2x$ BECOMES MAXIMUM OR MINIMUM?

Multiply this expression by 3, and let the product $= r$, $\therefore$ $x^3 - \frac{9}{2}x^2 + 6x - r = 0$, also let $a =$ one of the roots of this equation.

$$x-a \,\big)\, x^3 - \frac{9}{2}x^2 + 6x - r = 0 \,\big(\, x^2 + \left(a - \frac{9}{2}\right)x + a^2 - \frac{9a}{2} + 6 = 0, \text{(A)}$$

$$\begin{array}{l} x^3 - ax^2 \\ \hline \left(a - \frac{9}{2}\right)x^2 + 6x \\ \left(a - \frac{9}{2}\right)x^2 - a\left(a - \frac{9}{2}\right)x \\ \hline (a^2 - \frac{9a}{2} + 6)\,x - r \\ (a^2 - \frac{9a}{2} + 6)\,x - a(a^2 - \frac{9a}{2} + 6) \\ \hline \end{array}$$

$\therefore$ $r = a(a^2 - \frac{9a}{2} + 6)$ $\therefore$ $\frac{r}{a} = a^2 - \frac{9a}{2} + 6$. Now from equation (A) $x^2 + \left(a - \frac{9}{2}\right)x = -\frac{r}{a}$ or $x = -\frac{2a-9}{4} \pm \sqrt{\frac{\left(a - \frac{9}{2}\right)^2 \times a - 4r}{4a}}$ and in order that r or $4r$ may be a max. we must have $a\left(a - \frac{9}{2}\right)^2 = 4r = 4a\,(a^2 - \frac{9a}{2} + 6)$

or $a^2 - 9a + \frac{81}{4} = 4a^2 - 18a + 24$ or $3a^2 - 9a = -\frac{15}{4}$ $\therefore a^2 - 3a = -\frac{5}{4} \therefore a = \frac{3}{2} \pm 1 = \frac{5}{2}$ or $\frac{1}{2}$ and $x = -\frac{2a - 9}{4} = \frac{4}{4} = 1$ for maximum; $x = -\frac{2a-9}{4} = -\frac{1-9}{2} = \frac{8}{4} = 2$ for min.

The same solved without impossible roots.

In the equation $x^2 + \left(a - \frac{9}{2}\right)x = -\frac{r}{a}$ let $x = y - \frac{a - \frac{9}{2}}{2}$ $\therefore x^2 + \left(a - \frac{9}{2}\right)x = y^2 - \left(a - \frac{9}{2}\right)y + \frac{\left(a - \frac{9}{2}\right)^2}{4} + \left(a - \frac{9}{2}\right)y - \frac{\left(a - \frac{9}{2}\right)^2}{2} = y^2 - \frac{\left(a - \frac{9}{2}\right)}{4} = -\frac{r}{a}$

$r = \frac{a\left(a - \frac{9}{2}\right)^2}{4} - ay^2 =$ max. when $y = 0$, $\therefore r = \frac{a\left(a - \frac{9}{2}\right)^2}{4}$; but $r = a(a^2 - \frac{9a}{2} + 6)$ $\therefore \frac{a\left(a - \frac{9}{2}\right)^2}{4} = a(a^2 - \frac{9a}{2} + 6)$ or $a^2 - 3a = -\frac{5}{4}$ and $a = \frac{3}{2} \pm 1 = \frac{5}{2}$ or $\frac{1}{2}$ and $x = \frac{2a - 9}{4} = 2$ or 1 as before.

PROB. (18.) WHAT NUMBER IS THAT FROM THE CUBE OF WHICH ITS SQUARE AND TWENTY-ONE TIMES ITSELF BEING SUBTRACTED, THE REMAINDER IS THE GREATEST POSSIBLE?

Let x = number required; then according to the question $x^3 - x^2 - 21\,x = \text{max.} = r$ $\therefore$ $x^3 - x^2 - 21\,x - r = 0$. Also suppose a = one of the roots of this equation.

$$\begin{array}{l} x-a\,\big)\ x^3-x^2-21x-r=0\ \big(\,x^2+(a-1)x+a^2-a-21=0, \text{(A)} \\ \quad\ \ \underline{x^3-ax^2} \\ \qquad\ (a-1)\,x^2-21x \\ \qquad\ \underline{(a-1)\,x^2-a(a-1)\,x} \\ \qquad\qquad\quad (a^2-a-21)\,x-r \\ \qquad\qquad\quad \underline{(a^2-a-21)\,x-a(a^2-a-21)} \end{array}$$

$\therefore$ $r = a(a^2 - a - 21)$ $\therefore$ $\dfrac{r}{a} = a^2 - a - 21$ $\therefore$ from equation (A) $x^2 + (a-1)\,x = -\dfrac{r}{a}$ or $x = -\dfrac{a-1}{2} \pm \sqrt{\dfrac{a(a-1)^2 - 4r}{4a}}$.

Now in order that r or $4r$ may become a maximum we must have $a(a-1)^2 = 4r = 4a\,(a^2 - a - 21)$ or $a^2 - \dfrac{2}{3}\,a = \dfrac{85}{3}$ and $\therefore$ $a = -5$, $\therefore$ $x = -\dfrac{a-1}{2} = 3$.

The same solved without impossible roots.

In the equation $x^2 + (a-1)\,x = -\dfrac{r}{a}$ let $x = y - \dfrac{a-1}{2}$ and $\therefore$ $x^2 + (a-1)\,x = y^2 - (a-1)\,y + \dfrac{(a-1)^2}{4} + (a-1)\,y - \dfrac{(a-1)^2}{2} = y^2 - \dfrac{(a-1)^2}{4} = \dfrac{r}{a}$ $\therefore$ $r = \dfrac{a(a-1)^2}{4} - ay^2 = \text{max.}$ when $y = 0$, and $\therefore$ $r = \dfrac{a(a-1)^2}{4}$; but $r =$

$a(a^2 - a - 21) \therefore \frac{a(a-1)^2}{4} = a\ (a^2 - a - 21) \therefore a^2 - \frac{2}{3}a = \frac{85}{3}$ or $a = -\frac{15}{3} = -5$, and $x = -\frac{a-1}{2} = -\frac{-5-1}{2} = 3$ as before.

PROB. (19.) TO CUT THE GREATEST ELLIPSE FROM A GIVEN CONE. (Fig. 50.)

Let ABD be the cone, PB the elliptic section, $AC = a$, $Cn = x$, major axis $= 2m = PB$, $BC = b$, $nP = y$, minor axis $= 2n = ro$. Now the area of the Ellipse $= \pi mn$ (see the Integral Calculus, or my treatise called an Insight into the Nature of the Integral Calculus). It is evident that $PB : lB :: PQ : El$ or $2 : 1 :: PQ : El$, and $BP : Pl :: BD : lF$ or $2 : 1 :: BD : lF \therefore PQ = 2El$, and $BD = 2lF$ or $PQ \times BD = 4El \times lF = 4lo^2 = ro^2 = 4n^2 \therefore 2n = \sqrt{PQ \times BD} = \sqrt{2x \times 2b} = 2\sqrt{bx}$, and $2m = \sqrt{Bn^2 + Pn^2} = \sqrt{(b+x)^2 + Pn^2}$ but $Pn = CA \times \frac{Dn}{CD} = \frac{a(b-x)}{b} \therefore 2m = \sqrt{(b+x)^2 + \frac{a^2}{b^2}(b-x)^2}$ and $\therefore$ area $= \pi nm = \frac{\pi\sqrt{bx}}{2}\sqrt{(b+x)^2 + \frac{a^2}{b^2}(b-x)^2} =$ max. and $\therefore x(b+x)^2 + \frac{a^2}{b^2}x(b-x)^2 =$ max. and therefore $\frac{a^2+b^2}{b^2}x^3 - \frac{2(a^2-b^2)}{b}x^2 + (a^2+b^2)x =$ max. Dividing this expression by the constant quantity $\frac{a^2+b^2}{b^2}$ we have $x^3 - \frac{2b(a^2-b^2)}{a^2+b^2}x^2 + b^2x =$ max. To shorten the calculation let $\frac{2b(a^2-b^2)}{a^2+b^2} = q \therefore x^3 - qx^2 + b^2x =$ max. $= r$, $\therefore x^3 - qx^2 + b^2x - r = 0$. Now suppose $v =$ one of the roots of this equation.

$$\begin{array}{l} x-v\,\big)\; x^3-qx^2+b^2x-r=0\;\big(\,x^2+(v-q)x+v^2-vq+b^2=0\;(A) \\ \quad\;\; \underline{x^3-vx^2} \\ \quad\;\; (v-q)\,x^2+b^2x \\ \quad\;\; \underline{(v-q)\,x^2-v(v-q)\,x} \\ \qquad\qquad (v^2-vq+b^2)\,x-r \\ \qquad\qquad \underline{(v^2-vq+b^2)\,x-v(v^2-vq+b^2)} \end{array}$$

$\therefore\ r=v(v^2-vq+b^2)$ and $\dfrac{r}{v}=v^2-vq+b^2$. Now from equation (A) we find $x^2+(v-q)\,x=-\dfrac{r}{v}$ and $\therefore\ x=-\dfrac{v-q}{2}\pm\sqrt{\dfrac{v(v-q)^2-4r}{4v}}$, and in order that r or $4r$ may become $=$ max. we must have $v(v-q)^2=4r=4v\,(v^2-vq+b^2)$ or $v^2-\dfrac{2}{3}\,qv=\dfrac{q^2-4b^2}{3}\ \therefore\ v=\dfrac{q\pm\sqrt{4q^2-12b^2}}{3}$, and $x=-\dfrac{v-q}{2}=\dfrac{q-v}{2}=\dfrac{q}{2}-\dfrac{v}{2}$ $=\dfrac{3q}{2\times 3}-\dfrac{q\pm\sqrt{4q^2-12b^2}}{2\times 3}=\dfrac{2q\pm\sqrt{4q^2-12b^2}}{2\times 3}=\dfrac{q\pm\frac{1}{2}\sqrt{4q^2-12b^2}}{3}=\dfrac{q\pm\sqrt{q^2-3b^2}}{3}$; but $q=\dfrac{2b(a^2-b^2)}{a^2+b^2}$

$\therefore\ x=\dfrac{2b(a^2-b^2)\pm b\sqrt{a^4-14a^2b^2+b^4}}{3\,(a^2+b^2)}$. This problem is possible so long as the altitude a and base $2b$ are such as make $a^4-14a^2b^2+b^4$ a positive quantity. The limit of possibility is when the radical disappears; then we have the following equation $a^4-14b^2a^2+49b^4=48b^4\ \therefore\ a^2=7b^2\pm\sqrt{48b^4}=b^2(7\pm 4\sqrt{3})\ \therefore\ x=\dfrac{2b}{3}\cdot\dfrac{6\pm 4\sqrt{3}}{8\pm 4\sqrt{3}}=\dfrac{b}{3}\cdot\dfrac{3+2\sqrt{3}}{2+\sqrt{3}}$.

The same solved without impossible roots.

In the equation $x^2 + (v - q)\,x = -\dfrac{r}{v}$ let $x = y - \dfrac{v-q}{2}$,

$\therefore\ x^2 + (v - q)\,x = y^2 - (v - q)\,y + \dfrac{(v-q)^2}{4} + (v - q)\,y - \dfrac{(v-q)^2}{2} = y^2 - \dfrac{(v-q)^2}{4} = -\dfrac{r}{v}$ or $r = \dfrac{v(v-q)^2}{4} - vy^2 =$ max. when $y = 0$, $\therefore\ \dfrac{v(v-q)^2}{4} = r = v(v^2 - vq + b^2)$.

From this equation as before we may find $v = \dfrac{q \pm \sqrt{4q^2 - 12b^2}}{3}$ and hence $x = \dfrac{2b(a^2 - b^2) \pm b\sqrt{a^4 - 14a^2b^2 + b^4}}{3(a^2 + b^2)}$ as before.

PROB. (20.) THE CORNER OF A LEAF IS TURNED BACK, SO AS JUST TO REACH THE OTHER EDGE; FIND WHEN THE LENGTH OF THE CREASE IS A MINIMUM. (Fig. 51.)

The full leaf is *mnAB*, and when its corner A is turned back and touches the other edge mB of the page at the point a, the triangular piece QPA of the leaf falls upon its remaining piece $mBPQn$, and each of the angles QaP and QAP is $= 90°$, and consequently the figure $QaPA$ may be inscribed in a circle.

It is also evident that $aP = PA$ and $aQ = AQ$ and by the property of the circle $aA \times PQ = 2AQ \times AP$ .... (1.)

Now let $PA = x$ and $AB = a$ $\therefore$ by Prop. 12, 2nd book of Euclid $Aa^2 = aP^2 + AP^2 + 2BP \times PA = 2x^2 + 2(a - x)\,x = 2x^2 + 2ax - 2x^2 = 2ax$ $\therefore$ $Aa = \sqrt{2ax}$. Now $AQ^2 = QP^2 - AP^2$ $\therefore$ from equation (1) $aA^2 \times PQ^2 = 4AQ^2 \times AP^2 = 4AP^2 \times PQ^2 - 4AP^4$ $\therefore$ $4AP^4 = (4AP^2 - aA^2)\,PQ^2$ $\therefore$ $4x^4 = (4x^2 - 2ax)\,PQ$ $\therefore$ $PQ = \dfrac{2x^3}{2x - a} =$ min. $\therefore$ $\dfrac{2x - a}{2x^3}$

$=$ max. Let $2x - a = y$, $\therefore x = \dfrac{y+a}{2}$ and $2x^3 = \dfrac{(y+a)^3}{4}$

$\therefore \dfrac{2x-a}{2x^3} = \dfrac{4y}{(y+a)^3} = $ max. $\therefore \dfrac{y}{(y+a)^3}$. Now let $y =$

$$\frac{ab}{c} \therefore \frac{y}{(y+a)^3} = \frac{\frac{ab}{c}}{\frac{a^3(b+c)^3}{c^3}} = \frac{bc^2}{(c+b)^3} \times \frac{1}{a^2} = \text{max.},$$

$\dfrac{1}{a^2}$ is a constant given quantity, $\therefore \dfrac{bc^2}{(c+b)^3} = $ max. It is evident that $\dfrac{bc^2}{(c+b)^3} = \dfrac{b}{c+b} \times \dfrac{c^2}{(c+b)^2} = \left(1 - \dfrac{c}{c+b}\right) \times \dfrac{c^2}{(c+b)^2}$. Now let $\dfrac{c}{c+b} = z$, $\therefore \left(1 - \dfrac{c}{c+b}\right) \times \dfrac{c^2}{(c+b)^2}$ $= (1-z)\,z^2 = z^2 - z^3 = $ max. $\therefore$ by Prob. 2nd, $z = \dfrac{2}{3} = \dfrac{c}{c+b} \therefore \dfrac{3}{2} = \dfrac{c+b}{c} = 1 + \dfrac{b}{c}$ or $\dfrac{b}{c} = \dfrac{1}{2} \therefore y = \dfrac{ab}{c}$ $= \dfrac{a}{2}$ and $x = \dfrac{y+a}{2} = \dfrac{\frac{a}{2}+a}{2} = \dfrac{3a}{4}$.

The same may easily be solved without impossible roots.

PROB. (21.) TO FIND THE POSITION OF THE PLANET *Venus* IN RESPECT OF THE EARTH, WHEN HER LIGHT IS THE GREATEST. (Fig. 52.)

The planet does not appear brightest when her disc is perfectly round; she is then too remote to produce that effect; and besides, she is seen in the direction of the sun. In her inferior conjunction her crescent is too narrow, almost the whole illuminated part being turned towards the sun. It is therefore in some intermediate position, which is to be determined, that she is brightest. Let S be the Sun, E the Earth,

and *ABCD Venus, ABD* its illuminated hemisphere, which is turned towards the Sun, and *CBD* its hemisphere towards the Earth: produce *SV* to *F*.

The portion of the illuminated surface towards the Earth is contained between two planes *DV*, *BV*, perpendicular to the plane *EVS*; and this surface will manifestly be projected into a crescent, the breadth of which is the versed sine of the angle *BVD*, which is equal to *EVF*, because if *BVE* be added to both, each is a right angle.

Now the area of the crescent is always as its breadth; therefore, the whole disc being taken as a unit, the illuminated part will be expressed by the versed sine of the angle *EVF*, or by $1 + \cos. EVS$. Again the brightness of the planet is inversely as the square of the distance, therefore the brightness depending on its position, in respect of the Sun and its distance from the earth jointly, will be proportional to $\dfrac{1 + \cos. EVS}{EV^2}$. Let $a = ES$, the distance of the Earth from the Sun, $b = VS$ the distance of Venus from the Sun, $x = VE$, the distance of Venus from the Earth. Then $\cos. EVS = \dfrac{x^2 + b^2 - a^2}{2bx}$, and therefore the brightness of the planet $= \dfrac{1 + \cos. EVS}{EV^2} = \dfrac{x^2 + 2bx + b^2 - a^2}{2bx^3} =$ max. or $\dfrac{x^2 + 2bx + b^2 - a^2}{x^3} =$ max. which let $= r$, $\therefore x^2 + 2bx + b^2 - a^2 = rx^3$ or $rx^3 - x^2 - 2bx + a^2 - b^2 = 0$. Now let $x = \dfrac{1}{y}$ $\therefore \dfrac{r}{y^3} - \dfrac{1}{y^2} - \dfrac{2b}{y} + a^2 - b^2 = 0$, and $\therefore (a^2 - b^2)y^3 - 2by^2 - y + r = 0$, and dividing this equation by $a^2 - b^2$ we find $y^3 - \dfrac{2b}{a^2 - b^2}y^2 - \dfrac{1}{a^2 - b^2}y + \dfrac{r}{a^2 - b^2} = 0$.

Now since $r =$ max. and $\dfrac{1}{a^2 - b^2} =$ constant quantity, $\therefore$

$\frac{r}{a^2-b^2}$ = max. which let $= v$; also let $\frac{2b}{a^2-b^2} = m$ and $\frac{1}{a^2-b^2} = n$ ....(1)

$y^3 - my^2 - ny + v = 0$. Suppose that c = one of the negative roots of this equation, and consequently $y + c$ must exactly divide the said equation

$$
\begin{array}{l}
y+c \,\big)\, y^3 - my^2 - ny + v = 0 \,\big(\, y^2 - (c+m)y + c^2 + mc - n = 0 \text{ (A)} \\
\quad\quad\underline{y^3 + cy^2} \\
\quad\quad\quad - (c+m)\,y^2 - ny \\
\quad\quad\quad \underline{- (c+m)\,y^2 - c(c+m)\,y} \\
\quad\quad\quad\quad\quad\quad (c^2 + cm - n)\,y + v \\
\quad\quad\quad\quad\quad\quad \underline{(c^2 + cm - n)\,y + c(c^2 + cm - n)}
\end{array}
$$

$\therefore v = c(c^2 + cm - n)$, $\therefore \frac{v}{c} = c^2 + cm - n$, and from equation (A) we have $y^2 - (c+m)\,y = -\frac{v}{c}$ or $y = \frac{c+m}{2} \pm \sqrt{\frac{c(c+m)^2 - 4v}{4c}}$. Now in order that $4v$ or v may be = max. we must have $c(c+m)^2 = 4v = 4c\,(c^2 + cm - n)$ or $c^2 + 2cm + m^2 = 4c^2 + 4cm - 4n \therefore c^2 + \frac{2m}{3}c = \frac{m^2 + 4n}{3}$ or $c = -\frac{m}{3} + \sqrt{\frac{4m^2 + 12n}{9}} = \frac{2\sqrt{m^2 + 3n} - m}{3}$. Now $y = \frac{c+m}{2} = \frac{\sqrt{m^2 + 3n} + m}{3}$, and from equation (1) taking the values of m and n we find $\sqrt{m^2+3n} = \sqrt{\frac{4b^2 + 3a^2 - 3b^2}{(a^2-b^2)^2}}$ $= \frac{\sqrt{3a^2 + b^2}}{a^2 - b^2} \therefore \frac{\sqrt{m^2 + 3n} + m}{3} = \frac{\sqrt{3a^2 + b^2} + 2b}{3(a^2 - b^2)} =$ $\frac{\sqrt{3a^2 + b^2} + 2b}{3a^2 + b^2 - 4b^2} = \frac{1}{\sqrt{3a^2 + b^2} - 2b}$ and $\therefore \frac{1}{y} = x =$ $\sqrt{3a^2 + b^2} - 2b$.

In numbers $a = 10{,}000$, $b = 7{,}233$, therefore $x = 4{,}304$, the angles $E = 39^\circ\ 43'\ 30''$, $V = 117^\circ\ 55'\ 20''$, $S = 22^\circ\ 21'\ 10''$. —(From the 7th edition of the Encyclopædia Britannica.)

The same solved without impossible roots.

In the equation $y^2 - (c + m)\,y = -\frac{v}{c}$ let $y = z + \frac{c + m}{2}$
$\therefore y^2 - (c + m)\,y = z^2 + (c + m)\,z + \frac{(c + m)^2}{4} - (c + m)$
$z - \frac{(c + m)^2}{2} = z^2 - \frac{(c + m)^2}{4} = -\frac{v}{c}$ or $v = \frac{c(c + m)^2}{4}$
$- cz^2$, which is evidently a maximum when $z = 0$, $\therefore v = \frac{c(c + m)^2}{4}$. But $v = c\,(c^2 + cm - n)$ $\therefore \frac{c(c + m)^2}{4} = c\,(c^2 + cm - n)$ $\therefore c = \frac{2\sqrt{m^2 + 3n} - m}{3}$ and therefore $y = \frac{c + m}{2} = \frac{\sqrt{m^2 + 3n^2} + m}{3}$ as before.

PROB. (22.) REQUIRED TO DETERMINE WHAT MUST BE THE DIAMETER OF A WATER-WHEEL, SO AS TO RECEIVE THE GREATEST EFFECT FROM A STREAM OF WATER OF 12 FEET FALL. (Fig. 53.)

In the case of an undershot-wheel put the height of the water $AB = 12$ feet $= a$ and the radius BC or CD of the wheel $= x$, the water falling perpendicularly on the extremity of the radius CD at D. Then $AC = a - x$, and the velocity due to this height, or with which the water strikes the wheel at D will be as $\sqrt{a - x}$, because the squares of times or velocities are as the spaces, and consequently velocities are as the square roots of spaces, and therefore the effect on the wheel, being as the velocity and as the length of the lever CD, will be denoted by $x\sqrt{a - x}$ or $\sqrt{ax^2 - x^3}$, which therefore must

be a max. or its square $ax^2 - x^3 = \text{max.}$ Let $ax^2 - x^3 = r$ or $x^3 - ax^2 + r = 0$; also let $b =$ a negative root of this equation $\therefore x + b$ must exactly divide it.

$$x + b \,)\, x^3 - ax^2 + r = 0 \,(\, x^2 - (b + a)x + b^2 + ab = 0, \ldots \text{(A.)}$$

$$\begin{array}{l} x^3 + bx^2 \\ \hline -(b + a)\,x^2 + r \\ -(b + a)\,x^2 - b(b + a)\,x \\ \hline \qquad (b^2 + ab)\,x + r \\ \qquad (b^2 + ab)\,x + b(b^2 + ab) \\ \hline \end{array} \quad \therefore r = b\,(b^2 + ab)$$

$\therefore \frac{r}{b} = b^2 + ab$ $\therefore$ from equation (A) we find $x^2 - (b + a)\,x = -\frac{r}{b}$ $\therefore x = \frac{b + a}{2} \pm \sqrt{\frac{b(b + a)^2 - 4r}{4b}}$ which, when r or $4r = \text{max.}$ must give $b(b + a)^2 = 4r = 4b\,(b^2 + ab) = 4b^2\,(a + b)$ $\therefore b + a = 4b$ and $b = \frac{a}{3}$ $\therefore x = \frac{b + a}{2} = \frac{2a}{3}$; but $a = 12$, $\therefore x = \frac{12 \times 2}{3} = 8$ feet radius.

But if the water be considered as conducted so as to strike on the bottom of the wheel, as in the annexed figure (Fig. 54), it will then strike the wheel with its greatest velocity, and there can be no limit to the size of the wheel, since the greater the radius or lever *BC*, the greater will be the effect.—(From the 3rd vol. of the old edition of Hutton's Course of Mathematics.) In the case of an overshot-wheel $a - 2x$ will be the fall of water, $\sqrt{a - 2x}$ as the velocity, and $x\sqrt{a - 2x}$ or $\sqrt{ax^2 - 2x^3}$ the effect, then $ax^2 - 2x^3$ is a maximum. Here instead of x we must put down $2x$ $\therefore 2x = \frac{2a}{3}$ $\therefore x = \frac{a}{3} = 4$, the radius of the wheel.

But all these calculations are to be considered as independent of the resistance of the wheel, and of the weight of the water in the buckets of it.

The same solved without impossible roots.

In the equation $x^2 - (b + a)\,x = -\frac{r}{b}$, let $x = \frac{b + a}{2} + y$ $\therefore\ x^2 - (b + a)\,x = y^2 + (b + a)\,y + \frac{(b + a)^2}{4} - (b + a)$ $y - \frac{(b + a)^2}{2} = y^2 - \frac{(b + a)^2}{4} = -\frac{r}{b}\ \therefore\ r = \frac{b(b + a)^2}{4}$ $- by^2 =$ max. when $y = 0\ \therefore\ r = \frac{b(b + a)^2}{4}$. But $r = b(b^2 + ab)\ \therefore\ \frac{b(b + a)^2}{4} = b(b^2 + ab)$ and $4b = b + a\ \therefore$ $b = \frac{a}{3}$ and $x = \frac{b + a}{2} = \frac{2a}{3}$; but $a = 12\ \therefore\ x = \frac{12 \times 2}{3}$ $= 8$ feet as before.

PROB. (23.) TO DETERMINE THE STRONGEST ANGLE OF POSITION OF A PAIR OF GATES FOR THE LOCK ON A CANAL OR RIVER. (Fig. 55.)

Let AC, BC be the two gates, meeting in the angle C, projecting out against the pressure of the water, AB being the breadth of the canal or river. Now the pressure of water on a gate AC, is as the quantity or as the extent or length of it, AC, and the mechanical effect of that pressure, is as the length of lever to half AC, or to AC itself. On both these accounts then the pressure is as AC^2. Therefore the resistance or the strength of the gate must be as the reciprocal of AC^2. Now produce AC to meet BD, perpendicular to it, in D; and draw CE to bisect AB perpendicularly at E; then by similar triangles $AC : AE :: AB : AD$; where, AE and AB being given lengths, AD is reciprocally as AC, or AD^2 reciprocally as AC^2; that is, AD^2 is as the resistance of the gate AC. But the resistance of AC is increased by the pressure of the other gate in the direction

BC. Now the force in *BC* is resolved into the two *BD*, *DC*; the latter of which, *DC*, being parallel to *AC*, has no effect upon it, but the former, *BD*, acts perpendicularly on it. Therefore the whole effective strength or resistance of the gate is as the product $AD^2 \times BD$. If now there be put $AB = a$, and $BD = x$, then $AD^2 = AB^2 - BD^2 = a^2 - x^2$; consequently $AD^2 \times BD = (a^2 - x^2) \times x = a^2x - x^3$ for the resistance of either gate: and if we would have this to be the greatest, or the resistanee a maximum, we must find such a value of x which will make $a^2x - x^3 = \text{max.} = r$. Let $b =$ one of the negative roots of this equation, and consequently $x + b$ must divide it exactly.

$$x + b \,)\, x^3 - a^2x + r = 0 \,(\, x^2 - bx + b^2 - a^2 = 0, \ \ldots \ \text{(A.)}$$

$$\begin{array}{l}
x^3 + bx^2 \\
\hline
\quad - bx^2 - a^2x \\
\quad - bx^2 - b^2x \\
\hline
\qquad (b^2 - a^2)\, x + r \\
\qquad (b^2 - a^2)\, x + b(b_2 - a^2) \\
\hline
\end{array} \quad \therefore \ r = b(b^2 - a^2)$$

$\therefore \ \dfrac{r}{b} = b^2 - a^2 \ \therefore$ from equation (A) we find $x^2 - bx = -\dfrac{r}{b} \therefore x = \dfrac{b}{2} \pm \sqrt{\dfrac{b^3 - 4r}{4b}}$. Now in order that r or $4r$ may be $=$ max. we must have $b^3 = 4r = 4b\,(b^2 - a^2)$ or $b = \dfrac{2a}{\sqrt{3}}$ and $x = \dfrac{b}{2} = \dfrac{a}{\sqrt{3}} = a\sqrt{\dfrac{1}{3}} = \cdot 57735a$, the natural sine of 35° 16′; that is, the strongest position for the lock gates is when they make the angle *A* or *B* = 35° 16′; or the complemental angle *ACE* or *BCE* = 54° 44′, or the whole salient angle *ACB* = 109° 28′.—(From Hutton's Fluxions.)

The same solved without impossible roots.

In the equation $x^2 - bx = -\frac{r}{b}$ let $x = y + \frac{b}{2}$ and $\therefore$ $x^2 - bx = y^2 + by + \frac{b^2}{4} - by - \frac{b^2}{2} = y^2 - \frac{b^2}{4} = -\frac{r}{b}$ $\therefore r = \frac{b^3}{4} - by^2 =$ max. when $y = 0$ $\therefore r = \frac{b^3}{4}$; but $r = b(b^2 - a^2)$ $\therefore \frac{b^3}{4} = b(b^2 - a^2)$ and $\therefore b = \frac{2a}{\sqrt{3}}$ or $x = \frac{b}{2} = \frac{a}{\sqrt{8}}$ as before.

PROB. (24.) IT IS REQUIRED TO DETERMINE THE SIZE OF A CUBICAL SOLID, WHICH BEING LET FALL INTO A CONICAL VESSEL FULL OF WATER SHALL EXPEL THE MOST WATER POSSIBLE, FROM THE VESSEL; ITS DEPTH BEING $= a$ AND DIAMETER OF THE MOUTH $= 2b$. (Fig. 56.)

Let ABC be the given vessel, the diameter of its mouth $= 2b$ and its depth $HC = a$. $EmnD =$ the required cube. Let $FC = x$. Now by similar triangles we find $HC : AH :: FC : EF$ or $a : b :: x : EF$ or $EF = \frac{bx}{a}$ and $\therefore ED = 2EF = \frac{2bx}{a}$, and consequently the area of the base of the required cube $= \left(\frac{2bx}{a}\right)^2 = \frac{4b^2x^2}{a^2}$ which being multiplied by HF ($= HC - FC = a - x =$ the height of the immersed part of the cube) the product $= \frac{4b^2x^2}{a^2}(a - x) =$ the solid content of the immersed part of the cube $=$ quantity of water displaced. Now since $\frac{4b^2}{a^2}$ is a constant quantity, therefore $x^2(a - x) = ax^2 - x^3 = \text{max.} = r$ $\therefore x^3 - ax^2 + r = 0$.

Let c = one of the negative roots of this equation, consequently $x + c$ must exactly divide it.

$$x + c \,)\, x^3 - ax^2 + r = 0 \,(\, x^2 - (c + a)\,x + c^2 + ca \ldots (\text{A.})$$
$$x^3 + cx^2$$
$$\overline{}$$
$$- (a + c)\,x^2 + r$$
$$- (a + c)\,x^2 - c(c + a)\,x$$
$$\overline{}$$
$$c(c + a)\,x + r$$
$$c(c + a)\,x + c(c^2 + ca)$$

$\therefore\ r = c(c^2 + ca)\ \therefore\ \dfrac{r}{c} = c^2 + ca$. Now from equation (A) we have $x^2 - (a + c)\,x = -\dfrac{r}{c}\ \therefore\ x = \dfrac{a + c}{2} \pm \sqrt{\dfrac{c(a + c)^2 - 4r}{4c}}$, and in order that r or $4r$ may become a max. we must have $c(c + a)^2 = 4r = 4\,c(c^2 + ca) = 4c^2 (c + a)\ \therefore\ c + a = 4c$ and $c = \dfrac{a}{3}\ \therefore\ x = \dfrac{c + a}{2} = \dfrac{2a}{3}$ and consequently one of the equal sides of the required cube $=$

$$ED = \frac{2bx}{a} = \frac{2b \times \frac{2a}{3}}{3} = \frac{4ba}{9}.$$

The same solved without impossible roots.

In the equation $x^2 - (c + a)\,x = -\dfrac{r}{c}$ let $x = y + \dfrac{c + a}{2}$ $\therefore\ x^2 - (c + a)\,x = y^2 + (c + a)\,y + \dfrac{(c + a)^2}{4} - (c + a)\,y - \dfrac{(c + a)^2}{2} = y^2 - \dfrac{(c + a)^2}{4} = -\dfrac{r}{c}\ \therefore\ r = \dfrac{c(c + a)^2}{4} - cy^2 = \text{max. when } y = 0\ \therefore\ \dfrac{c(c + a)^2}{4} = r = c(c^2 + ca)\ \therefore$ $c + a = 4c$ or $c = \dfrac{a}{3}\ \therefore\ x = \dfrac{c + a}{2} = \dfrac{2a}{3}$ as before

PROB. (25.) IT IS REQUIRED TO DETERMINE THE SIZE OF A BALL, WHICH, BEING LET FALL INTO A CONICAL VESSEL FULL OF WATER, SHALL EXPEL THE MOST WATER POSSIBLE FROM THE VESSEL; ITS DEPTH BEING 6 AND DIAMETER 5 INCHES. (Fig. 57.)

Let ABC represent the cone of the vessel, and DHE the ball, touching the sides in the points D and E, the centre of the ball being at some point F in the axis of the cone. Put $AG = GB = 2\frac{1}{2} = a$, $GC = 6 = b$ $\therefore$ $AC = \sqrt{AG^2 + GC^2} = 6\frac{1}{2} = c$, $DF = FE = FH = x$ the radius of the ball. The two triangles ACG and DCF are equiangular; therefore $AG : AC :: DF : FC$; that is $a : c :: x : \frac{cx}{a} = FC$; hence $GF = GC - FC = b - \frac{cx}{a}$ and $GH = GF + FH = b + x - \frac{cx}{a}$ = height of the segment immersed in the water. Then (by Hutton's and other authors' works on Geometry,—see Introduction,) the content of the immersed segment will be $(6\,DF - 2GH) \times GH^2 \times \cdot 5236 = \left(6x - 2x - 2b + \frac{2cx}{a}\right) \times \left(x + b - \frac{cx}{a}\right)^2 \times \cdot 5236$ = maximum, and therefore $\left(2x - b + \frac{cx}{a}\right)\left(x + b - \frac{cx}{a}\right)^2$ = max.; but $2x - b + \frac{cx}{a} = \frac{2a + c}{a}x - b$ and $x + b - \frac{cx}{a} = \frac{a - c}{a}x + b = b - \frac{c - a}{a}x$ where c is greater than a, because c is the hypothenuse and a the perpendicular of a right-angled triangle. Let $b - \frac{c - a}{a}x = y$ $\therefore$ $x = \frac{(b - y)\,a}{c - a}$ and consequently $\frac{2a + c}{a}x - b = \frac{(b - y)\,a\,(2a + c)}{a(c - a)} - b = \frac{3a^2b - a(2a + c)\,y}{a(c - a)} =$

$\frac{3ab - (2a + c)y}{c - a}$ $\therefore$ $\left(\frac{2a + c}{a}x - b\right)\left(\frac{a - c}{a}x + b\right)^2 =$ $\left(\frac{2a + c}{a}x - b\right)\left(b - \frac{c - a}{ax}\right)^2 = \frac{3aby^2 - (2a + c)y^3}{c - a} =$ $\frac{2a + c}{c - a}\left(\frac{3ab}{2a + c}y^2 - y^3\right) = \max.$ Now as $\frac{2a + c}{c - a} =$ a constant quantity, we must also have $\frac{3ab}{2a + c}y^2 - y^3 = \max.$ $= r$; also let $\frac{3ab}{2a + c} = A$, and $\therefore y^3 - Ay^2 + r = 0$. Let $n =$ one of the negative roots of this equation;

$$
\begin{array}{l}
y+n \,)\; y^3 - Ay^2 + r = 0 \;(\, y^2 - (n + A)\,y + n(n + A) = 0, \text{ (B.)} \\
\quad \underline{y^3 + ny^2} \\
\qquad - (n + A)\,y^2 + r \\
\qquad \underline{- (n + A)\,y^2 - n(n + A)\,y} \\
\qquad\qquad\qquad n(n + A)\,y + r \\
\qquad\qquad\qquad \underline{n(n + A)\,y + n^2(n + A)} \;\therefore r = n^2
\end{array}
$$

$(n + A)$ and $\frac{r}{n} = n(n + A)$ $\therefore$ from equation (B) we have $y^2 - (n + A)\,y = -\frac{r}{n}$ or $y = \frac{n + A}{2} \pm \sqrt{\frac{n(n + A)^2 - 4r}{4n}}$ and hence it is evident that $4r$ cannot be greater than $n(n + A)^2$ and therefore when it is a max. we must have $n(n + A)^2 = 4r$ $= 4n^2(n + A)$ and $\therefore n + A = 4n$ or $n = \frac{A}{3}$; and hence $y = \frac{n + A}{2} = \frac{2A}{3} = \frac{2 \times 3ab}{3(2a + c)} = \frac{2ab}{2a + c}$ and $x =$

$$\frac{(b - y)\,a}{c - a} = \frac{\left(b - \frac{2ab}{2a + c}\right) \times a}{c - a} = \frac{abc}{(c - a)(2a + c)} = 2\tfrac{11}{92},$$

the radius of the ball; consequently its diameter is $4\frac{11}{46}$ inches, as required.

The same solved without impossible roots.

In the equation $y^2 - (n + A)\,y = -\frac{r}{n}$ let $y = z + \frac{n + A}{2}$ $\therefore\ y^2 - (n + A)\,y = z^2 + (n + A)\,z + \frac{(n + A)^2}{4} - (n + A)\,z - \frac{(n + A)^2}{2} = z^2 - \frac{(n + A)^2}{4} = -\frac{r}{n}$ $\therefore\ r = \frac{n(n + A)^2}{4} - nz^2 =$ max. when $z = 0$ $\therefore\ r = \frac{n(n + A)^2}{4}$, but $r = n^2\,(n + A)$ $\therefore\ n^2\,(n + A) = \frac{n(n + A)^2}{4}$ $\therefore\ n = \frac{A}{3}$.

Also $y = \frac{n + A}{2} = \frac{2A}{3}$ and $x = \frac{(b - y)\,a}{c - a} = \frac{\left(b - \frac{2A}{3}\right)\,a}{c - a}$; but $A = \frac{3ab}{2a + c}$ $\therefore\ x = \frac{abc}{(c - a)\,(2a + c)}$ as before.

PROB. (26.) TO FIND SUCH A VALUE OF x AS SHALL MAKE $\frac{(x - 1)^2}{(x + 1)^3}$ A MAXIMUM.

Let $x + 1 = \frac{1}{y}$ $\therefore\ (x + 1)^3 = \frac{1}{y^3}$, $x - 1 = \frac{1}{y} - 2 = \frac{1 - 2y}{y}$ and $(x - 1)^2 = \frac{(1 - 2y)^2}{y^2}$ and therefore we find $\frac{(x - 1)^2}{(x + 1)^3} = \frac{(1 - 2y)^2}{y^2} \times \frac{y^3}{1} = (1 - 2y)^2 \times y = y - 4y^2 + 4y^3 = 4\,(y^3 - y^2 + \frac{1}{4}y) =$ max. and $\therefore\ y^3 - y^2 + \frac{1}{4}y =$ max. Now let $y = z + \frac{1}{3}$

$$\begin{aligned} \therefore\ y^3 &= z^3 + z^2 + \frac{1}{3}z + \frac{1}{27} \\ -\,y^2 &= -\,z^2 - \frac{2}{3}z - \frac{1}{9} \\ \frac{1}{4}y &= \qquad\quad + \frac{1}{4}z + \frac{1}{12} \end{aligned}$$

$\therefore y^3 - y^2 + \frac{1}{4}y = z^3 - \frac{1}{12}z + \frac{1}{12} - \frac{2}{27} =$ max. and $\frac{1}{12} - \frac{2}{27}$ is a constant quantity, and $\therefore z^3 - \frac{1}{12}z =$ max. $= r$, $\therefore z^3 - \frac{1}{12}z - r = 0$.

Let one of the positive roots of this equation $= a$, and consequently $z - a$ must exactly divide it.

$$z - a \,\big)\, z^3 - \frac{1}{12}z - r = 0 \,\big(\, z^2 + az + a^2 - \frac{1}{12} = 0, \ldots \text{(A.)}$$

$$z^3 - az^2$$

$$az^2 - \frac{1}{12}z$$

$$az^2 - a^2z$$

$$\left(a^2 - \frac{1}{12}\right)z - r$$

$$\left(a^2 - \frac{1}{12}\right)z - a\left(a^2 - \frac{1}{12}\right)$$

$$\therefore r = a\left(a^2 - \frac{1}{12}\right)$$

and $a^2 - \frac{1}{12} = \frac{r}{a}$ $\therefore$ from equation (A) $z^2 + az = -\frac{r}{a}$ or $z = -\frac{a}{2} \pm \sqrt{\frac{a^3 - 4r}{4a}}$ where $4r$ cannot be greater than a^3, $\therefore$ when $r =$ max. we must have $a^3 = 4r = 4a\left(a^2 - \frac{1}{12}\right)$ or $a^2 = 4a^2 - \frac{1}{3}$ $\therefore a^2 = \frac{1}{9}$ and $a = \frac{1}{3}$. Also $z = -\frac{a}{2}$ $= -\frac{1}{6}$ and $y = z + \frac{1}{3} = \frac{1}{6}$, $\therefore x + 1 = \frac{1}{y} = 6$, $\therefore x = 5$.

The same solved without eliminating the second term of the cubic equation $y^3 - y^2 + \frac{1}{4}y - r = 0$.

Let $a =$ one of the positive roots of this equation, and consequently $y - a$ must exactly divide it.

$$y-a \,\big)\, y^3-y^2+\frac{1}{4}y-r=0 \,\big(\, y^2+(a-1)y+\left(a-\frac{1}{2}\right)^2=0, \text{ (A.)}$$

$$\underline{y^3-ay^2}$$

$$(a-1)\,y^2+\frac{1}{4}y$$

$$\underline{(a-1)\,y^2-a(a-1)\,y}$$

$$\left(a-\frac{1}{2}\right)^2 y-r$$

$$\underline{\left(a-\frac{1}{2}\right)^2 y-a\left(a-\frac{1}{2}\right)^2}$$

$\therefore\ r = a\left(a-\frac{1}{2}\right)^2$ or $\frac{r}{a} = \left(a-\frac{1}{2}\right)^2$, and from equation (A) $y^2+(a-1)\,y = -\frac{r}{a}$ or $y = -\frac{a-1}{2} \pm \sqrt{\frac{a(a-1)^2-4r}{4a}}$.

Now in order that r or $4r$ may become a max. we must have $a(a-1)^2 = 4r = 4a\left(a-\frac{1}{2}\right)^2 \therefore a^2-2a+1 = 4a^2 - 4a+1$ or $a = \frac{2}{3}$ and $y = -\frac{a-1}{2} = \frac{1}{6} \therefore x+1 = \frac{1}{y} = 6$ and $x = 5$ as before.

The same solved without impossible roots.

In the equation $z^2+az = -\frac{r}{a}$ let $z = w-\frac{a}{2}$ and $\therefore$ $z^2+az = w^2-aw+\frac{a^2}{4}+aw-\frac{a^2}{2} = w^2-\frac{a^2}{4} = -\frac{r}{a}$ $\therefore r = \frac{a^3}{4} - aw^2 =$ max. when $w = 0 \therefore \frac{a^3}{4} = r = a\left(a^2-\frac{1}{12}\right) \therefore a = \frac{1}{3} \therefore z = -\frac{1}{6}$ and $y = z+\frac{1}{3} = \frac{1}{6}$ $\therefore x = \frac{1}{y} - 1 = 5$ as before.

PROB. (27.) TO SAW OUT OF THE TRUNK OF A TREE A RECTANGULAR BEAM THAT SHALL HAVE THE GREATEST POSSIBLE POWER OF SUSPENSION. (Fig. 58.)

Actual experiments lead to this result, that in a parallelopipedon of uniform thickness, supported on two points and loaded in the middle, the lateral strength is directly as the product of the breadth into the square of the depth, and inversely as the length.

Let $ACBm$ be the circumference of the trunk and the rectangle AB the base or top of the beam cut out of the trunk. $AB =$ diameter of the trunk $= a$, $AC =$ breadth $= x$, and $BC =$ depth of the beam $= \sqrt{a^2 - x^2}$. Also let $f =$ strength of the wood of which the tree is composed, and $l =$ the length of the beam which is in this problem $=$ a constant quantity. We have before observed that the power of suspension $=$ $\dfrac{f \times \text{breadth} \times \text{depth}^2}{\text{length}} = \dfrac{fx(a^2 - x^2)}{l} = \dfrac{f}{l}(a^2x - x^3) =$ max. $\therefore a^2x - x^3 = \text{max.} = r \therefore x^3 - a^2x + r = 0$. Let one of the negative roots of this equation $= b$, and consequently $x + b$ must exactly divide it.

$$
\begin{array}{l}
x + b \,)\; x^3 - a^2x + r = 0 \;(\, x^2 - bx + b^2 - a^2 = 0, \;\ldots \text{(A.)} \\
\qquad\quad \underline{x^3 + bx^2} \\
\qquad\qquad - bx^2 - a^2x \\
\qquad\qquad \underline{- bx^2 - b^2x} \\
\qquad\qquad\qquad (b^2 - a^2)\,x + r \\
\qquad\qquad\qquad \underline{(b^2 - a^2)\,x + b(b^2 - a^2)} \quad \therefore r = b(b^2 - a^2)
\end{array}
$$

$\therefore b^2 - a^2 = \dfrac{r}{b}$. From equation (A) we find $x^2 - bx = -\dfrac{r}{b}$ or $x = \dfrac{b}{2} \pm \sqrt{\dfrac{b^3 - 4r}{4b}}$ and hence it is evident that when r or $4r =$ max., $b^3 = 4r = 4b\,(b^2 - a^2)$ or $3b^2 = 4a^2 \therefore b =$

$\frac{2a}{\sqrt{3}} \therefore x = \frac{b}{2} = \frac{a}{\sqrt{3}}$ = breadth and $\sqrt{a^2 - x^2} = \sqrt{a^2 - \frac{a^2}{3}}$ $= a\sqrt{\frac{2}{3}}$ = depth of the beam. Now from the points m and C draw mr and Cn perpendiculars to the diameter AB, then by prop. 8, 6th Book Euclid, we have $AB : AC :: AC : An$ or $a : x :: x : An = \frac{x^2}{a} = \frac{a^2}{3a} = \frac{a}{3}$. Also $AB : Bm :: Bm :$ Br or $a : x :: x : \frac{x^2}{a} = Br = \frac{a}{3} \therefore nr = AB - rB - An =$ $a - \frac{a}{3} - \frac{a}{3} = \frac{a}{3}$; hence the following construction.

Divide the diameter of the trunk into three equal parts, and from the two points of section draw the perpendiculars and complete the rectangle, which will be the base or top of the rectangular beam required.

The same solved without impossible roots.

In the equation $x^2 - bx = -\frac{r}{b}$ let $x = y + \frac{b}{2} \therefore x^2 - bx = y^2 + by + \frac{b^2}{4} - by - \frac{b^2}{2} = y^2 - \frac{b^2}{4} = -\frac{r}{b} \therefore r = \frac{b^3}{4} - by^2$ = max. when $y = 0 \therefore r = \frac{b^3}{4}$ or $b(b^2 - a^2) = \frac{b^3}{4}$ or $b = \frac{2a}{\sqrt{3}}$ and $x = \frac{b}{2} = \frac{a}{\sqrt{3}}$ as before.

CHAPTER III.

PROBLEMS OF MAXIMA AND MINIMA IN THE SOLUTIONS OF WHICH EQUATIONS OF THE FOURTH, FIFTH, SIXTH, AND SEVENTH DEGREE ARE USED.

Section 1.

PROB. (1.) WHAT FRACTION IS THAT THE FOURTH POWER OF WHICH BEING SUBTRACTED FROM ITS CUBE THE REMAINDER IS THE GREATEST POSSIBLE?

Let x = the fraction required, $\therefore x^3 - x^4 = \text{max.} = r$ $\therefore x^4 - x^3 + r = 0$. Now let the product of the two values of this equation $= x^2 - ax + b$, which must consequently divide it exactly, and $\therefore$ we find,

$$x^2-ax+b \;)\; x^4-x^3+r=0 \;(\; x^2+(a-1)x+a^2-a-b=0, \ldots (1)$$

$$\begin{array}{l}
x^4-ax^3+bx^2 \\
\hline
(a-1)\,x^3 - bx^2 + r \\
(a-1)\,x^3 - a(a-1)\,x^2 + b(a-1)\,x \\
\hline
(a^2 - a - b)\,x^2 - b(a-1)\,x + r \\
(a^2-a-b)x^2-a(a^2-a-b)x+b(a^2-a-b.) \\
\hline
\end{array}$$

Now it has been proved in the introductory chapter that when any equation is divided by two factors of the form $x-c$, $x-d$, successively, or by their product of the form $x^2 - ax + b$ at once, then the remainder R must be equal to zero and entirely independent of x in the case when c and d are supposed to be the roots of the given equation. We therefore find $b(a-1) = a(a^2 - a - b)$ and $r = b(a^2 - a - b)\ldots(2)$;

$\therefore \dfrac{r}{b} = a^2 - a - b$. Also we have $b(a-1) = a(a^2 - a - b)$

or $ab - b = a^3 - a^2 - ab$, $\therefore 2ab - b = b(2a - 1) = a^3 - a^2 \therefore b = \frac{a^3 - a^2}{2a - 1} = \frac{a^2(a-1)}{2a-1} \therefore a^2 - a - b = a^2 - a - \frac{a^3 - a^2}{2a-1} = \frac{a(a-1)^2}{(2a-1)} \therefore r = b(a^2 - a - b) = \frac{a^2(a-1)}{2a-1} \times \frac{a(a-1)^2}{2a-1} = \frac{a^3(a-1)^3}{(2a-1)^2}$ and $\therefore 4r = \frac{4a^3(a-1)^3}{(2a-1)^2}$ ...... (3).

Now from equation (1) we find, $x^2 + (a-1)\,x = -\frac{r}{b}$ and solving this quadratic we find $x = -\frac{a-1}{2} \pm \sqrt{\frac{b(a-1)^2 - 4r}{4b}}$

$$= -\frac{a-1}{2} \pm \sqrt{\frac{\frac{(a-1)^2\,a^2(a-1)}{2a-1} - \frac{4a^3(a-1)^3}{(2a-1)^2}}{4b}}.$$ Here it is evident that $4r$ or $\frac{4a^3(a-1)^3}{(2a-1)^2}$ cannot be taken so great as to make it greater than $b(a-1)^2$ or $\frac{(a-1)^2\,a^2(a-1)}{2a-1}$ and consequently when $r = \max.$ we must have $\frac{(a-1)^2 a^2(a-1)}{2a-1} = \frac{4a^3(a-1)^3}{(2a-1)^2}$ or $1 = \frac{4a}{2a-1}$ or $2a - 1 = 4a \therefore a = -\frac{1}{2}$ and $x = -\frac{a-1}{2} = -\frac{\frac{1}{2}-1}{2} = \frac{3}{4}$.

The same solved without impossible roots.

In the equation $x^2 + (a-1)\,x = -\frac{r}{b}$ let $x = y - \frac{a-1}{2}$ and $\therefore x^2 + (a-1)\,x = y^2 - (a-1)\,y + \frac{(a-1)^2}{4} + (a-1)\,y - \frac{(a-1)^2}{2} = y^2 - \frac{(a-1)^2}{4} = -\frac{r}{b}$ and $r = \frac{b(a-1)^2}{4} - by^2$, which is evidently a maximum when $y = 0$, and consequently $r = \frac{b(a-1)^2}{4}$; but r is also $= \frac{a^3(a-1)^3}{(2a-1)^2}$

and therefore we find $\frac{b(a-1)^2}{4} = \frac{a^3(a-1)^3}{(2a-1)^3}$ or $b = \frac{4a^3(a-1)}{(2a-1)^2}$ or $\frac{a^2(a-1)}{2a-1} = \frac{4a^3(a-1)}{(2a-1)^2}$ $\therefore 1 = \frac{4a}{2a-1}$ or $a = -\frac{1}{2}$ $\therefore x = -\frac{a-1}{2} = -\frac{\frac{1}{2}-1}{2} = \frac{3}{4}$ as before.

PROB. (2.) TO FIND SUCH A FRACTION, THE FOURTH POWER OF WHICH BEING SUBTRACTED FROM ITSELF, LEAVES THE GREATEST REMAINDER POSSIBLE.

Let x = fraction required, then by the problem we find $x - x^4$ = max. which let $= r$ $\therefore x^4 - x + r = 0$. Let $x^2 - ax + b$ be the product of the two values of this equation, which must consequently be exactly divided by it,

$$\therefore x^2 - ax + b \,)\; x^4 - x + r = 0 \;(\; x^2 + ax + a^2 - b = 0, \ldots (1)$$

$$\begin{array}{l} x^4 - ax^3 + bx^2 \\ \hline \quad ax^3 - bx^2 - x + r \\ \quad ax^3 - a^2x^2 + abx \\ \hline \qquad (a^2 - b)\, x^2 - (ab + 1)\, x + r \\ \qquad (a^2 - b)\, x^2 - a\,(a^2 - b)\, x + b\,(a^2 - b) \\ \hline \end{array}$$

$\therefore ab + 1 = a^3 - ab$ $\therefore b = \frac{a^3 - 1}{2a}$, and $a^3 - b = a^2 - \frac{a^3 - 1}{2a} = \frac{a^3 + 1}{2a}$: and $r = b(a^2 - b) = \frac{a^3 - 1}{2a} \times \frac{a^3 + 1}{2a}$ $\frac{(a^3 - 1)\,(a^3 + 1)}{4a^2}$. Now from equation (1) we find $x^2 + ax = -\frac{r}{b}$ $\therefore x = -\frac{a}{2} \pm \sqrt{\frac{a^2b - 4r}{4b}}$. Hence it is evident that $4r$ cannot be greater than a^2b and $\therefore$ when r or $4r$ = max. we must have $a^2b = 4r$; but $b = \frac{a^3 - 1}{2a}$ and $\frac{(a^3 - 1)\,(a^3 + 1)}{4a}$

$= r \therefore \frac{a^2(a^3-1)}{2a} = \frac{4(a^3-1)(a^3+1)}{4a^2}$ or $\frac{2a^3(a^3-1)}{4a^2} = \frac{4(a^3-1)(a^3+1)}{4a^2} \therefore a^3 = 2a^3 + 2 \therefore a = \sqrt[3]{-2} = -\sqrt[3]{2}$

$\therefore r = -\frac{a}{2} = \frac{\sqrt[3]{2}}{2} = \sqrt[3]{\frac{2}{8}} = \frac{1}{\sqrt[3]{4}}.$

The same may be solved without impossible roots.

In the equation $x^2 + ax = -\frac{r}{b}$ let $x = y - \frac{a}{2}$ and therefore we find $x^2 + ax = y^2 - ay + \frac{a^2}{4} + ay - \frac{a^2}{2} = y^2 - \frac{a^2}{4} = -\frac{r}{b}$ or $r = \frac{a^2b}{4} - by^2$ which is evidently a maximum when $y = 0 \therefore r = \frac{a^2b}{4}$ or $4r = a^2b$. But $4r = \frac{4(a^3-1)(a^3+1)}{4a^2} \therefore a = -\sqrt[3]{2}$ and $x = -\frac{a}{2} = \frac{\sqrt[3]{2}}{2}$ $\sqrt[3]{\frac{2}{8}} = \frac{1}{\sqrt[3]{4}}$ as before.

PROB. (3.) TO DESCRIBE THE LEAST TRIANGLE TCt ABOUT A GIVEN PARABOLIC ARC APB OF WHICH C IS THE FOCUS. (Fig. 59.)

Let $AN = x$, $AC = a$, and therefore $tC = a + x$. Also by similar triangles we find, $tN : NP :: tC : CT$ or $2x : 2\sqrt{ax} :: a + x : CT = \frac{(a+x) \times \sqrt{a} \times \sqrt{x}}{x} \therefore \frac{CT}{2} = \frac{(a+x)\sqrt{a}}{2\sqrt{x}}$ and therefore the area $tTC = \frac{CT \times tC}{2} = \frac{(a+x)^2 \times \sqrt{a}}{2\sqrt{x}}$

$= \text{min. and} \therefore \dfrac{(a+x)^4}{x} = \text{min. and} \therefore \dfrac{x}{(a+x)^4} = \text{max.}$ Let $x = \dfrac{ab}{c}$ and $\therefore \dfrac{x}{(a+x)^4} = \dfrac{\frac{ab}{c}}{\frac{(ab+ac)^4}{c^4}} = \dfrac{abc^3}{a^4(c+b)^4} = \dfrac{1}{a^3} \times \dfrac{bc^3}{(c+b)^4}$ *or* $\dfrac{bc^3}{(c+b)^4} = \text{max.}$ It is evident that $\dfrac{bc^3}{(c+b)^4} = \dfrac{b}{c+a} \times \dfrac{c^3}{(c+b)^3} = \left(1 - \dfrac{c}{c+b}\right) \times \dfrac{c^3}{(c+b)^3}$. Let $y = \dfrac{c}{c+b} \therefore \dfrac{bc^3}{(c+b)^4} = (1-y)\,y^3 = y^3 - y^4 = \text{max.}$ In this case by problem (1) we find $y = \dfrac{3}{4}$, but $\dfrac{c}{c+b} = y = \dfrac{3}{4}$ *or* $\dfrac{c+b}{c} = \dfrac{4}{3} \therefore 1 + \dfrac{b}{c} = \dfrac{4}{3}$. But $x = \dfrac{ab}{c} \therefore \dfrac{x}{a} = \dfrac{b}{c}$ $\therefore 1 + \dfrac{x}{a} = \dfrac{4}{3} = 1 + \dfrac{1}{3} \therefore \dfrac{x}{a} = \dfrac{1}{3}$ and $x = \dfrac{a}{3}$.

The same may be solved without impossible roots as problem first.

The same may be solved by the following more direct and common way by which the two first problems have been solved.

Let $a + x = \dfrac{1}{y} \therefore$ the $\dfrac{x}{(a+x)^4} = \dfrac{\frac{1-ay}{y}}{\frac{1}{y^4}} = \dfrac{1-ay}{y} \times \dfrac{y^4}{1}$ $= y^3 - ay^4 = \text{max.}$ and $\therefore \dfrac{1}{a}y^3 - y^4 = \text{max.}$ which let $= r \therefore y^4 - \dfrac{1}{a}y^3 + r = 0$. Let $y^2 - by + c =$ product of the factors of the two values of this equation and consequently we have

$$y^2 - by + c \,\big)\, y^4 - \frac{1}{a}y^3 + r = 0 \,\big(\, y^2 + \left(b - \frac{1}{a}\right)y + b^2 - \frac{b}{a} - c = 0, \text{(1.)}$$

$$\underline{y^4 - by^3 + cy^2}$$

$$\left(b - \frac{1}{a}\right)y^3 - cy^2 + r$$

$$\frac{\left(b-\frac{1}{a}\right)y^3 - b\left(b-\frac{1}{a}\right)y^2 + c\left(b-\frac{1}{a}\right)y}{\left(b^2-\frac{b}{a}-c\right)y^2 - c\left(b-\frac{1}{a}\right)y + r}$$

$$\frac{\left(b^2-\frac{b}{a}-c\right)y^2 - b\left(b^2-\frac{b}{a}-c\right)y + c\left(b^2-\frac{b}{a}-c\right)}{}$$

and therefore $r = c\left(b^2 - \frac{b}{a} - c\right)$........ .... (2.)

and $c\left(b - \frac{1}{a}\right) = b\left(b^2 - \frac{b}{a} - c\right)$ ... . (3.)

From equation (3) we find $bc - \frac{c}{a} = b^3 - \frac{b^2}{a} - bc$ and consequently $2bc - \frac{c}{a} = b^3 - \frac{b^2}{a}$, $\therefore \frac{c(2ab-1)}{a} = \frac{ab^3 - b^2}{a}$

$\therefore c = \frac{b^2(ab-1)}{2ab-1}$ and from equation (2) we find $r =$

$c\left(b^2 - \frac{b}{a} - c\right) = \frac{b^2(ab-1)}{2ab-1} \times \left(b^2 - \frac{b}{a} - \frac{b^2(ab-1)}{2ab-1}\right)$

$= \frac{b^2(ab-1)}{2ab-1}\left(\frac{ab^2-b}{a} - \frac{b^2(ab-1)}{2ab-1}\right) = \frac{b^2(ab-1)}{2ab-1}$

$\left(\frac{a^2b^3 - 2ab^2 + b}{2a^2b - a}\right) = \frac{b^2(ab-1)}{2ab-1}\,\frac{(ab-1)^2 \times b}{2a^2b-a} = \frac{(ab-1)^3b^3}{(2ab-1)^2a}.$

From equation (1), we have $y^2 + \left(b - \frac{1}{a}\right)y + \frac{r}{c} = 0$, $\therefore$

$y = -\frac{b-\frac{1}{a}}{2} + \sqrt{\frac{\left(b-\frac{1}{a}\right)^2 \times c - 4r}{4c}}$ and hence it is evident that r or $4r$ cannot be taken so great as to become greater than $\left(b - \frac{1}{a}\right)^2 \times c$ and consequently when $r =$ max. we must have $4r = c\left(b - \frac{1}{a}\right)^2$. But $4r = \frac{4(ab-1)^3b^3}{(2ab-1)^2a}$

$\therefore\ 4r = c\left(b - \frac{1}{a}\right)^2 = \frac{b^2(ab-1)}{2ab-1}\left(b - \frac{1}{a}\right)^2 = \frac{b^2(ab-1)^3}{a^2(2ab-1)}$

$= \frac{4(ab-1)^3b^3}{(2ab-1)^2a}\ \therefore\ \frac{4b}{2ab-1} = \frac{1}{a},\ \therefore\ 2ab - 1 = 4ab$

and $\therefore\ 2ab = -1\ \therefore\ b = -\frac{1}{2a}$; but $y = -\dfrac{b - \frac{1}{a}}{2}$

$- \dfrac{-\frac{1}{2a} - \frac{1}{a}}{2} = \frac{3}{4a}\ \therefore\ a + x = \frac{1}{y} = \frac{4a}{3} = a + \frac{a}{3}$ and $\therefore$

$x = \frac{a}{3}$ as before.

The same may now easily be solved without impossible roots.

In the equation $y^2 + \left(b - \frac{1}{a}\right)y = -\frac{r}{c}$ let $y = z - \dfrac{b - \frac{1}{a}}{2}$ and $\therefore\ y^2 + \left(b - \frac{1}{a}\right)y = z^2 - \left(b - \frac{1}{a}\right)z + \dfrac{\left(b - \frac{1}{a}\right)^2}{4} + \left(b - \frac{1}{a}\right)z - \dfrac{\left(b - \frac{1}{a}\right)^2}{2} = z^2 - \dfrac{\left(b - \frac{1}{a}\right)^2}{4}$ $= -\frac{r}{c}$ and consequently $r = \dfrac{c\left(b - \frac{1}{a}\right)^2}{4} - cz^2$ which is evidently a max. when $z = 0$; $\therefore\ r = \dfrac{c\left(b - \frac{1}{a}\right)^2}{4}\ \therefore\ 4r =$ $c\left(b - \frac{1}{a}\right)^2 = \frac{c(ab-1)^2}{a^2}$ but $c = \frac{b^2(ab-1)}{2ab-1}\ \therefore\ 4r =$ $\frac{b^2(ab-1)^3}{a^2(2ab-1)}$ and $4r$ is also $= \frac{4(ab-1)^3b^3}{(2ab-1)^2 \times a}\ \therefore\ \frac{b^2(ab-1)^3}{a^2(2ab-1)}$ $= \frac{4(ab-1)^3 \times b^3}{(2ab-1)^2 \times a}\ \therefore\ \frac{1}{a} = \frac{4b}{2ab-1}\ \therefore\ b = -\frac{1}{2a}$, but y

$$= -\frac{b - \frac{1}{a}}{2} = -\frac{-\frac{1}{2a} - \frac{1}{a}}{2} = \frac{3}{4a} \text{ and } a + x = \frac{4a}{3} =$$

$a + \frac{a}{3}$ $\therefore x = \frac{a}{3}$ as before.

PROB. (4.) LET AB BE THE DIAMETER OF A CIRCLE, IT IS REQUIRED TO FIND A POINT, C, IN THE DIAMETER, SO THAT THE RECTANGLE FORMED BY THE CHORD DE, WHICH IS PERPENDICULAR TO AB, AND THE PART AC MAY BE THE GREATEST POSSIBLE. (Fig. 60.)

Let $AB = a$, $AC = x$, and $CB = a - x$, then $(a - x)x = CD^2$ and $CD = \sqrt{ax - x^2}$; therefore $DE = 2\sqrt{ax - x^2}$, and the rectangle $EG = x \times 2\sqrt{ax - x^2} =$ max. $\because$ its square $4x^2(ax - x^2)$ or $4ax^3 - 4x^4 =$ max. $\therefore ax^3 - x^4 =$ max. which let $= r$ $\therefore x^4 - ax^3 + r = 0$. Let $x^2 - bx + c =$ product of the two values of this equation, and therefore we find;

$$\begin{array}{l}
x^2 - bx + c \,\big)\, x^4 - ax^3 + r = 0 \,\big(\, x^2 + (b-a)x + b^2 - ab - c = 0, \text{ (A.)} \\
\qquad\qquad\quad x^4 - bx^3 + cx^2 \\
\hline
\qquad\qquad\quad (b - a)\,x^3 - cx^2 + r \\
\qquad\qquad\quad (b - a)\,x^3 - b(b - a)\,x^2 + c(b - a)\,x \\
\hline
\qquad\qquad\qquad (b^2 - ab - c)\,x^2 - c(b - a)\,x + r \\
\qquad\qquad\qquad (b^2 - ab - c)\,x^2 - b(b^2 - ab - c)\,x + c(b^2 - ab - c) \\
\hline
\end{array}$$

$\therefore c(b - a) = b(b^2 - ab - c)$ .... (1.)

and $r = c(b^2 - ab - c)$ $\therefore b^2 - ab - c = \frac{r}{c}$ .... (2.)

From equation (1), $c = \frac{b^2(b - a)}{2b - a}$ $\therefore b^2 - ab - c = b^2 - ab - \frac{b^2(b - a)}{2b - a} = \frac{b(b - a)^2}{2b - a} = \frac{r}{c}$. Now from...........(A.)

we have $x^2 + (b - a)x = -\frac{r}{c}$ or $x = \frac{b-a}{2} \pm \sqrt{\frac{(b-a)^2}{4} - \frac{r}{c}}$ and it is here evident that when r or $\frac{r}{c} =$ max. we must have $\frac{(b-a)^2}{4} = \frac{r}{c} = \frac{b(b-a)^2}{2b-a}$ $\therefore \frac{1}{4} = \frac{b}{2b-a}$ $\therefore b = -\frac{a}{2}$ and $x = -\frac{b-a}{2} = -\frac{-\frac{a}{2} - a}{2} = \frac{3a}{4}$.

The same solved without impossible roots.

In the equation $x^2 + (b-a)x = -\frac{r}{c}$ let $x = y - \frac{b-a}{2}$ $\therefore x^2 + (b-a)x = y^2 - (b-a)y + \frac{(b-a)^2}{4} + (b-a)y - \frac{(b-a)^2}{2} = y^2 - \frac{(b-a)^2}{4} = -\frac{r}{c}$ $\therefore r = \frac{c(b-a)^2}{4} - cy^2 =$ max. when $r = \frac{c(b-a)^2}{4}$ $\therefore \frac{(b-a)^2}{4} = \frac{r}{c} = \frac{b(b-a)^2}{2b-a}$ $\therefore \frac{1}{4} = \frac{b}{2b-a}$ or $b = -\frac{a}{2}$ and $x = -\frac{b-a}{2} = \frac{3a}{4}$ as before.

PROB. (5.) TO DIVIDE 12 INTO TWO PARTS, SO THAT THE LEAST MULTIPLIED BY THE CUBE OF THE GREATEST, SHALL BE A MAXIMUM.

Let $x =$ greater part $\therefore 12 - x =$ lesser part and $12x^3 - x^4 =$ max. $= r$ $\therefore x^4 - 12x^3 + r = 0$. Let the product of the two values of this equation $= x^2 - ax + b$.

$$\therefore x^2-ax+b \,)\, x^4-12x^3+r=0 \,(\, x^2+(a-12)x+a^2-12a-b=0, \text{(A.)}$$
$$\begin{array}{l} x^4 - ax^3 + bx^2 \\ \hline (a-12)\,x^3 - bx^2 \\ (a-12)\,x^3 - a(a-12)\,x^2 + b(a-12)\,x \\ \hline (a^2-12a-b)\,x^2 - b(a-12)\,x + r \\ (a^2-12a-b)x^2 - a(a^2-12a-b)x + b(a^2-12a-b) \\ \hline \end{array}$$

$\therefore\ r = b(a^2 - 12a - b)\ \therefore\ \frac{r}{b} = a^2 - 12a - b$.(1)

Also $b(a-12) = a(a^2 - 12a - b) = a^3 - 12a^2 - ab\ \therefore$ $b(2a-12) = a^3 - 12a^2\ \therefore\ b = \frac{a^3 - 12a^2}{2a-12} = \frac{a^2(a-12)}{2a-12}$

$\therefore$ from (1) $\frac{r}{b} = a^2 - 12a - b = a^2 - 12a - \frac{a^2(a-12)}{2a-12} =$ $\frac{a^3 - 24a^2 + 144a}{2a-12} = \frac{a(a-12)^2}{2a-12}$ and $r = b \times \frac{a(a-12)^2}{2a-12} =$ $\frac{a^2(a-12)}{2a-12} \times \frac{a(a-12)^2}{2a-12} = \frac{a^3(a-12)^3}{(2a-12)^2}$. Now from equation (A) $x^2 + (a-12)\,x = -\frac{r}{b}$ or $x = -\frac{a-12}{2} \pm$ $\sqrt{\frac{b(a-12)^2 - 4r}{4b}}$. Here it is evident that when r or $4r =$ max. we must have $b(a-12)^2 = 4r$ or $\frac{a^2(a-12)}{2a-12}(a-12)^2$ $= \frac{4a^3(a-12)^2}{(2a-12)^2}$ or $1 = \frac{4a}{2a-12}\ \therefore\ a = -6$ and $x = -\frac{a-12}{2} = 9$.

The same may be solved without impossible roots.

In the equation $x^2 + (a-12)\,x = -\frac{r}{b}$ let $x = y - \frac{a-12}{2}$ $\therefore\ x^2 + (a-12)\,x = y^2 - (a-12)\,y +$

$\frac{(a-12)^2}{4} + (a-12)\,y - \frac{(a-12)^2}{2} = y^2 - \frac{(a-12)^2}{4} = -\frac{r}{b} \therefore r = \frac{b(a-12)^2}{4} - by^2$ but $r = \frac{a^3(a-12)^3}{(2a-12)^2}$ and $b = \frac{a^2(a-12)}{2a-12} \therefore \frac{a^2(a-12)}{4(2a-12)}(a-12)^2 = \frac{a^3(a-12)^3}{(2a-12)^2} \therefore$ $a = -6$ and $x = -\frac{a-12}{2} = 9$ as before.

PROB. (6.) TO INSCRIBE THE GREATEST ISOSCELES TRIANGLE IN A GIVEN CIRCLE. (Fig. 61.)

Let ABC be the isosceles triangle required, and suppose $BD = x$ and $BE =$ diameter $= 2a \therefore DE = 2a - x \therefore$ half the base $= AD = \sqrt{2ax - x^2}$ and area of the isosceles triangle $= AD \times BD = x\sqrt{2ax - x^2} = \sqrt{2ax^3 - x^4} =$ max. $\therefore 2ax^3 - x^4 =$ max. $= r \therefore x^4 - 2ax^3 + r = 0$. $\therefore$ let $x^2 - bx + c =$ product of the two values of this equation,

$$x^2-bx+c)\ x^4-2ax^3+r=0\ (x^2+(b-2a)x+b^2-2ab-c=0,\ \text{(A.)}$$

$$\begin{array}{l}
x^4 - bx^3 + cx^2 \\
\hline
(b-2a)\,x^3 - cx^2 \\
(b-2a)\,x^3 - b(b-2a)\,x^2 + c(b-2a)x \\
\hline
(b^2-2ab-c)\,x^2 - c(b-2a)\,x + r \\
(b^2-2ab-c)x^2 - b(b^2-2ab-c)x + c(b^2-2ab-c) \\
\hline
\end{array}$$

$\therefore r = c(b^2 - 2ab - c) \therefore \frac{r}{c} = b^2 - 2ab - c$ .... (1.)

Also $c(b-2a) = b^3 - 2ab^2 - bc \therefore c = \frac{b^2(b-2a)}{2b-2a} \therefore$ $\frac{r}{c} = b^2 - 2ab - c = b^2 - 2ab - \frac{b^3 - 2ab^2}{2b - 2a} = \frac{b(b-2a)^2}{2b-2a}$

and $r = c \times \frac{b(b-2a)^2}{2b-2a} = \frac{b^2(b-2a)}{2b-2a} \times \frac{b(b-2a)^2}{2b-2a} = \frac{b^3(b-2a)^3}{(2b-2a)^2}$ or $4r = \frac{4b^3(b-2a)^3}{(2b-2a)^2}$. Now from equation (A) $x^2 + (b-2a)\,x = -\frac{r}{c}$ or $x = -\frac{b-2a}{2} \pm \sqrt{\frac{c(b-2a)^2 - 4r}{4c}}$ and here it is evident that when r or $4r$ $=$ max. we must have $c(b-2a)^2 = 4r$ or $\frac{b^2(b-2a)^3}{2b-2a} = \frac{4b^3(b-2a)^3}{(2b-2a)^2}$ $\therefore 1 = \frac{4b}{2b-2a}$ $\therefore b = -a$ and $x = -\frac{b-2a}{2} = \frac{3a}{2}$. Hence $AD = \sqrt{2ax - x^2} = \sqrt{3a^2 - \frac{9a^2}{4}}$ $= \frac{a\sqrt{3}}{2}$ $\therefore AC = 2AD = a\sqrt{3}$. $AB = \sqrt{AD^2 + BD^2}$ $\sqrt{\frac{3a^2}{4} + \frac{9a^2}{4}} = \sqrt{3a^2} = a\sqrt{3}$ $\therefore$ the triangle required is equilateral.

The same solved without impossible roots.

In the equation $x^2 + (b-2a)\,x = -\frac{r}{c}$ let $x = y - \frac{b-2a}{2}$ $\therefore x^2 + (b-2a)\,x = y^2 - (b-2a)\,y + \frac{(b-2a)^2}{4} + (b-2a)\,y - \frac{(b-2a)^2}{2} = y^2 - \frac{(b-2a)^2}{4} = -\frac{r}{c}$ $\therefore r = \frac{c(b-2a)^2}{4} - cy^2 =$ max. when $y = 0$ $\therefore r = \frac{c(b-2a)^2}{4}$. But $r = \frac{b^3(b-2a)^3}{(2b-2a)^2}$ and $c = \frac{b^2(b-2a)}{2b-2a}$ $\therefore \frac{b^3(b-2a)^3}{(2b-2a)^2} = \frac{b^2(b-2a)}{2b-2a} \times \frac{(b-2a)^2}{4}$ or $1 = \frac{4b}{2b-2a}$ $\therefore b = -a$ and

$x = -\frac{b-2a}{2} = \frac{3a}{2}$. Hence $AD = \sqrt{2ax - x^2} = \sqrt{3a^2 - \frac{9a^2}{4}} = \frac{a\sqrt{3}}{2}$ $\therefore AC = 2AD = 2 \times \frac{a\sqrt{3}}{2} = a\sqrt{3}$ as before.

PROB. (7.) TO INSCRIBE THE GREATEST PARABOLA IN A GIVEN ISOSCELES TRIANGLE. (Fig. 62.)

Let AGF be the given isosceles triangle and $CHPME$ the required parabola. Let $AD = b$, $GD = a$, and $GP = x$. Now KPG being a subtangent to the parabola, we must have by conic sections $GP = PK = x$ $\therefore$ $GK = 2x$; also $PK : PD :: HK^2 : CD^2$ .... (A.)

Now by similar triangles $GD : AD :: GK : HK$, or $a : b :: 2x : \frac{2bx}{a} = HK$ $\therefore$ by proportion (A), $x : a - x :: \frac{4b^2x^2}{a^2} : CD^2$ $\therefore$ $CD^2 = \frac{4b^2}{a^2}(a-x)x$ $\therefore$ $CD = \frac{2b}{a}\sqrt{(a-x)x}$. Now the area of the parabola or $\frac{2}{3} PD \times CD = \frac{2b}{a} \times \frac{2}{3}(a-x)\sqrt{(a-x)x} = \frac{4b}{3a}\sqrt{(a-x)^3x} = \text{max.}$ or $(a-x)^3x = \text{max.}$ Let $a - x = y$ $\therefore$ $x = a - y$ $\therefore$ $(a-x)^3x = y^3(a-y) = ay^3 - y^4 = \text{max.} = r$ $\therefore$ $y^4 - ay^3 + r = 0$. Proceeding exactly as in the solution of Prob. (4) we shall find $y = \frac{3a}{4}$ and $x = a - y = a - \frac{3a}{4} = \frac{a}{4}$.

The same may be solved without impossible roots as Prob. (4) was.

This problem if solved by the common method given in works on Diff. Calc. must ultimately produce a cubic equation, to solve which is generally tedious.

PROB. (8.) TO DETERMINE THE GREATEST PARABOLA THAT CAN BE FORMED BY CUTTING A GIVEN CONE ACD. (Fig. 63.)

Let nv, parallel to CA, be the axis of the parabola rvm and rm the base (or ordinate) thereof. Putting $DC = a$, $CA = b$, and $Dn = x$; then, by parallels, $a : b :: x : \frac{bx}{a} = nv$; moreover by the property of the circle, we have $rn^2 = nm^2 = Dn \times Cn = ax - x^2$, the square root of which multiplied by $\frac{2}{3} \times \frac{bx}{a}$ (because every parabola is $\frac{2}{3}$ of a parallelogram of the same base and altitude) gives $\frac{2bx}{3a}\sqrt{ax - x^2}$ for the contents of the parabola = max. $\therefore ax^3 - x^4 =$ max. $= r \therefore x^4 - ax^3 + r = 0$. Now by proceeding exactly as in Prob. (4) we find $x = \frac{3a}{4}$ when $ax^3 - x^4 =$ max.

The same may be solved without impossible roots in exactly the same manner in which Prob. (4) was.

PROB. (9.) THE CORNER OF A LEAF IS TURNED BACK, SO AS JUST TO REACH THE OTHER EDGE OF THE PAGE, FIND WHEN THE PART TURNED DOWN IS A MINIMUM. (See Fig. 51.)

It has been shown in Problem (20) Chapter 2nd that $aA \times PQ = 2AQ \times AP$ and that $aA = \sqrt{2ax}$, $PQ = \sqrt{\frac{2x^3}{2x - a}}$ $AP = x \therefore \sqrt{\frac{4ax^4}{2x - a}} = 2x \times AQ \therefore$ the area of the part

turned down $= \frac{x \times AQ}{2} = \frac{2}{4}\sqrt{a}\sqrt{\frac{x^4}{2x-a}} = \frac{\sqrt{a}}{2}$ $\sqrt{\frac{x^4}{2x-a}} = \text{min.} \therefore \frac{x^4}{2x-a} = \text{min.}$ Let $2x - a = y \therefore$ $\frac{y+a}{2} = x \therefore \frac{x^4}{2x-a} = \frac{\frac{(y+a)^4}{16}}{y} = \frac{(y+a)^4}{16y} = \text{min.}$ or $\frac{16y}{(y+a)^4} = \text{max.}$ or $\frac{y}{(y+a)^4} = \text{max.}$ Also let $y = \frac{ab}{c}$ $\therefore \frac{\frac{ab}{c}}{(y+a)^4} = \frac{\frac{ab}{c}}{\frac{a^4(b+c)^4}{c^4}} = \frac{c^4ab}{a^4c(b+c)^4} = \frac{c^3b}{a^3(b+c)^4} =$ $\frac{1}{a^3} \times \frac{bc^3}{(b+c)^4} = \text{max.}$ $\frac{bc^3}{(b+c)^4} = \text{max.}$ But $\frac{bc^3}{(b+c)^4}$ $= \left(1 - \frac{c}{b+c}\right)\frac{c^3}{(b+c)^3}$; let $\frac{c}{b+c} = z \therefore (1-z)z^3 =$ $z^3 - z^4 = \text{max.}$ Proceeding exactly as in problem (4) we find $z = \frac{3}{4}$ or $\frac{c}{b+c} = \frac{3}{4} \therefore \frac{b+c}{c} = \frac{b}{c} + 1 = \frac{4}{3} =$ $\frac{1}{3} + 1 \therefore \frac{b}{c} = \frac{1}{3}$ and $\frac{ab}{c} = \frac{a}{3} = y$ and $x = \frac{y+a}{2} =$ $\frac{\frac{a}{3}+a}{2} = \frac{2a}{3}$.

The same may be solved without impossible roots as Prob. (4) was.

Section 2.

PROB. (10.) TO FIND SUCH A VALUE OF x AS MAY MAKE $mx^4 - x^5 = \text{MAX.} = r$.

We have now the equation $x^5 - mx^4 + r = 0$, and let the product of the three values of this equation $= x^3 + ax^2 + bx + c$ and $\therefore$ we have

$$x^3+ax^2+bx+c \,)\; x^5-mx^4+r=0 \;(\, x^2-(a+m)x+a^2+am-b=0, (1)$$

$$\begin{array}{l}
x^5+ax^4+bx^3+cx^2 \\
\hline
-(a+m)x^4-bx^3-cx^2 \\
-(a+m)x^4-(a^2+am)x^3-(ab+bm)x^2-(ac+cm)x \\
\hline
(a^2+am-b)x^3+(ab+bm-c)x^2+(ac+cm)x+r \\
(a^2+am-b)x^3+(a^3+a^2m-ab)x^2+(a^2b+abm-b^2)x \\
+c(a^2+am-b) \\
\hline
\end{array}$$

$\therefore\ a^2 + am - b = \dfrac{r}{c}$ .... (2.)

Also $ab + bm - c = a^3 + a^2m - ab$ .... (3.)

$ac + cm = a^2b + abm - b^2$ or $c = \dfrac{a^2b + abm - b^2}{a+m}$... (4.)

From (3) and (4) we have $ab + bm - c = ab + bm - \dfrac{a^2b + abm - b^2}{a+m} = \dfrac{a^2b + abm + abm + bm^2 - a^2b - abm + b^2}{a+m}$ $= \dfrac{abm + bm^2 + b^2}{a+m} = a^3 + a^2m - ab$, or $mab + bm^2 + b^2 = a^4 + 2ma^3 + a^2m^2 - a^2b - mab$ or $b^2 + (2ma + m^2 + a^2)\, b = (a^2 + am)^2$ or $b^2 + (a+m)^2 b = a^2\,(a+m)^2$ or $b = -\dfrac{(a+m)^2}{2} + \sqrt{\dfrac{(a+m)^4 + 4a^2\,(a+m)^2}{4}}$ or $b = -\dfrac{(a+m)^2 + \sqrt{(a+m)^4 + 4a^2\,(a+m)^2}}{2}$.

From (1) $x^2 - (a+m)\,x = -\dfrac{r}{c}$ or $x = \dfrac{a+m}{2} + \sqrt{\dfrac{(a+m)^2}{4} - \dfrac{r}{c}}$ $\therefore$ when $r = \max.\ \dfrac{(a+m)^2}{4} = \dfrac{r}{c} = a^2 + am - b = \dfrac{2a(a+m) + (a+m)^2 - (a+m)\sqrt{(a+m)^2 + 4a^2}}{2}$ or $a + m = 4a + 2a + 2m - 2\sqrt{(a+m)^2 + 4a^2}$ or $5a + m = 2\sqrt{(a+m)^2 + 4a^2}$ or $25a^2 + 10am + m^2 = 4a^2 + 8am + 4m^2 + 16a^2$ or $5a^2 + 2am = 3m^2$, $\therefore\ a^2 + \dfrac{2m}{5}\, a = \dfrac{3m^2}{5}$

$\therefore a = -\frac{m}{5} \pm \sqrt{\frac{15m^2}{25} + \frac{m^2}{25}} = \frac{4m}{5} - \frac{m}{5} = \frac{3m}{5}$, and

$x = \frac{a + m}{2} = \frac{\frac{3m}{5} + m}{2} = \frac{4m}{5}$. If $m = 1$, then $x = \frac{4}{5}$.

PROB. (11.) TO FIND SUCH A VALUE OF x AS MAY MAKE $mx^3 - x^5 =$ MAX. $= r$.

We have now the equation $x^5 - mx^3 + r = 0$, and let the product of the three values of this equation $= x^3 + ax^2 + bx + c$, we therefore find,—

$$
\begin{array}{l}
x^3+ax^2+bx+c \,\big)\; x^5-mx^3+r=0 \;\big(\, x^2-ax+a^2-b-m=0, \ldots\; \text{(A.)} \\
\qquad\qquad\qquad\quad x^5 + ax^4 + bx^3 + cx^2 \\
\qquad\qquad\qquad\quad \overline{- ax^4 - (b + m)\, x^3 - cx^2} \\
\qquad\qquad\qquad\quad - ax^4 - a^2x^3 - abx^2 - cax \\
\qquad\qquad\qquad\quad \overline{(a^2-b-m)\, x^3+(ab-c)\, x^2+cax+r} \\
\qquad\qquad\qquad\quad (a^2-b-m)x^3+(a^3-ab-am)x^2+(a^2b-b^2-bm)x \\
\qquad\qquad\qquad\quad +ca^2-bc-cm
\end{array}
$$

$$\therefore r = ca^2 - bc - cm \text{ or } a^2 - b - m = \frac{r}{c} \ldots\ldots\ldots\ldots\ldots \text{(1.)}$$

$$ab - c = a^3 - ab - am \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.)}$$

$$ca = a^2b - b^2 - bm \text{ or } c = \frac{a^2b - b^2 - bm}{a} \quad \ldots \text{(3.)}$$

and $\therefore ab - c = ab - \frac{a^2b - b^2 - bm}{a} = \frac{b^2 + bm}{a}$ $a^3 - ab - am$ $\therefore b^2 + bm = a^4 - a^2b - a^2m$ or $b = -\frac{a^2 + m}{2} \pm \sqrt{\frac{a^4 + 2a^2m + m^2 + 4a^4 - 4a^2m}{4}} = -\frac{a^2 + m}{2} - \sqrt{\frac{5a^4 - 2a^2m + m^2}{4}} = -\frac{(a^2 + m) + \sqrt{5a^4 - 2a^2m + m^2}}{2}$

and equation (A) gives $x^2 - ax = -\frac{r}{c} \therefore x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - \frac{r}{c}}$

$\therefore \dfrac{a^2}{4} = \dfrac{r}{c} = a^2 - m - b = \dfrac{2a^2 - 2m + a^2 + m - \sqrt{5a^4 - 2a^2m + m^2}}{2}$ and $a^2 = 6a^2 - 2m - 2\sqrt{5a^4 - 2a^2m + m^2}$ or $5a^2 - 2m = 2\sqrt{5a^4 - 2a^2m + m^2}$ $\therefore$ $25a^4 - 20a^2m + 4m^2 = 20a^4 - 8a^2m + 4m^2$ or $5a^4 - 12ma^2 = 0$ $\therefore$ $a^2 = \dfrac{12m}{5}$, and $x = \dfrac{a}{2}$ $\therefore$ $x^2 = \dfrac{a^2}{4} = \dfrac{12m}{4 \times 5} = \dfrac{3m}{5}$ $\therefore$ $x = \sqrt{\dfrac{3m}{5}}$. If $m = 1$, then $x = \sqrt{\dfrac{8}{5}}$.

PROB. (12.) TO FIND SUCH A VALUE OF x AS MAY MAKE $mx^2 - x^5 = \text{MAX.} = r$.

We have the equation $x^5 - mx^2 + r = 0$, and let the product of three values of this equation $= x^3 + ax^2 + bx + c$ $\therefore$

$$
\begin{array}{l}
x^3 + ax^2 + bx + c \,\big)\; x^5 - mx^2 + r = 0 \;\big(\, x^2 - ax + a^2 - b = 0, \ldots (1.)\\
\qquad\qquad\qquad\quad \underline{x^5 + ax^4 + bx^3 + cx^2}\\
\qquad\qquad\qquad\quad - ax^4 - bx^3 - (c + m)\, x^2\\
\qquad\qquad\qquad\quad \underline{- ax^4 - a^2x^3 - abx^2 - acx}\\
\qquad\qquad\qquad\qquad (a^2 - b)\, x^3 + (ab - c - m)x^2 + acx + r\\
\qquad\qquad\qquad\qquad \underline{(a^2 - b)x^3 + (a^3 - ab)x^2 + (a^2b - b^2)x + c(a^2 - b)}
\end{array}
$$

$\therefore a^2 - b = \dfrac{r}{c}$ .... (2.)

$ab - m - c = a^3 - ab$ .... (3.)

$ac = a^2b - b^2$ or $c = \dfrac{a^2b - b^2}{a}$ .... (4.)

$\therefore ab - m - c = ab - m - \dfrac{a^2b - b^2}{a} = \dfrac{a^2b - am - a^2b + b^2}{a} = \dfrac{b^2 - am}{a} = a^3 - ab$ or $b^2 - am = a^4 - a^2b$ or $b^2 + a^2b = a^4 + am$ $\therefore$ $b = -\dfrac{a^2}{2} + \sqrt{\dfrac{a^4}{4} + a^4 + am} = \dfrac{-a^2 + \sqrt{5a^4 + 4am}}{2}$ $\therefore$ $a^2 - b = \dfrac{3a^2 - \sqrt{5a^4 + 4am}}{2}$ and from (1)

and (2) $x = \frac{a}{2} + \sqrt{\frac{a^2}{4} - \frac{r}{c}}$ $\therefore$, when $r =$ max. we must have $\frac{a^2}{4} = \frac{r}{c} = a^2 - b = \frac{3a^2 - \sqrt{5a^4+4am}}{2}$ $\therefore a^2 = 6a^2 - 2\sqrt{5a^4 + 4am}$ $\therefore 25a^4 = 20a^4 + 16am$ $\therefore a^3 = \frac{16m}{5}$, and $x = \frac{a}{2}$ $\therefore x^3 = \frac{a^3}{8} = \frac{2m}{5}$ $\therefore x = \sqrt[3]{\frac{2m}{5}}$. If $m = 1$, then $x = \sqrt[3]{\frac{2}{5}}$.

PROB. (13.) TO FIND SUCH A VALUE OF x AS MAY MAKE $mx - x^5 = \text{MAX} = r$.

We have $x^5 - mx + r = 0$, and let the product of three values of this equation $= x^3 + ax^2 + bx + c$, and therefore we have,

$$
\begin{array}{l}
x^3+ax^2+bx+c \,\big)\; x^5-mx+r=0 \;\big(\, x^2-ax+a^2-b=0, \ldots. \text{(1.)} \\
\qquad\qquad\qquad\quad \underline{x^5 + ax^4 + bx^3 + cx^2} \\
\qquad\qquad\qquad\quad - ax^4 - bx^3 - cx^2 - mx \\
\qquad\qquad\qquad\quad \underline{- ax^4 - a^2x^3 - abx^2 - acx} \\
\qquad\qquad\qquad\quad (a^2-b)x^3+(ab-c)x^2+(ac-m)x+r \\
\qquad\qquad\qquad\quad \underline{(a^2-b)x^3+(a^3-ab)x^2+(a^2b-b^2)x+c(a^2-b)}
\end{array}
$$

$\therefore c(a^2 - b) = r$ $\therefore a^2 - b = \frac{r}{c}$ .... (2.)

Also $ab - c = a^3 - ab$ .... (3.)

$ca - m = a^2b - b^2$ $\therefore c = \frac{a^2b - b^2 + m}{a}$ .... (4.)

$\therefore ab - c = ab - \frac{a^2b - b^2 + m}{a} = \frac{b^2 - m}{a} = a^3 - ab$ or $b^2 - m = a^4 - a^2b$ $\therefore b^2 + a^2b = a^4 + m$ and $\therefore b = \frac{-a^2 + \sqrt{5a^4 + 4m}}{2}$. Now from (1), $x = \frac{a}{2} + \sqrt{\frac{a^2}{4} - \frac{r}{c}}$,

$\therefore$ when $r = \max.,\ \dfrac{a^2}{4} = \dfrac{r}{c} = a^2 - b = a^2 -$

$= \dfrac{3a^2 - \sqrt{5a^4 + 4m}}{2}\ \therefore\ a^2 = 6a^2 - 2\sqrt{5a^4 + 4m}\ \therefore\ 25a^4$

$= 20a^4 + 16m,\ \therefore\ a^4 = \dfrac{16m}{5}.$ Now since $\dfrac{a^2}{4} = \dfrac{r}{c}$ we must

have $x = \dfrac{a}{2},\ \therefore\ x^4 = \dfrac{a^4}{16} = \dfrac{16m}{16 \times 5} = \dfrac{m}{5},\ \therefore\ x = \sqrt[4]{\dfrac{m}{5}}.$

If $m = 1$, then $x = \sqrt[4]{\dfrac{1}{5}} = \dfrac{1}{\sqrt[4]{5}}.$

Section 3.

PROB. (14.) TO FIND SUCH A VALUE OF x AS MAY MAKE $mx^5 - x^6 = \text{MAX.} = r$.

Since we have $x^6 - mx^5 + r = 0$, let the product of four values of this equation $= x^4 + ax^3 + bx^2 + cx + d\ \therefore$

$x^4 + ax^3 + bx^2 + cx + d\ )\ x^6 - mx^5 + r = 0\ (\ x^2 - (a+m)\,x + a^2 + am - b = 0,$ (1.)

$$
\begin{array}{l}
x^6 + ax^5 + bx^4 + cx^3 + dx^2 \\
\hline
-\ (a+m)\,x^5 - bx^4 - cx^3 - dx^2 \\
-\ (a+m)x^5 - (a^2 + am)\,x^4 - (ab + bm)\,x^3 \\
-\ (ca + cm)\,x^2 - (ad + dm)\,x \\
\hline
(a^2 + am - b)\,x^4 + (ab + bm - c)\,x^3 + (ca + cm - d)\,x^2 + (ad + dm)\,x \\
(a^2 + am - b)x^4 + (a^3 + a^2m - ab)\,x^3 + (a^2b + abm - b^2)\,x^2 + \\
(ca^2 + cam - bc)x \\
\hline
+\ r \\
+\ d\,(a^2 + am - b) \\
\hline
\end{array}
$$

$\therefore\ r = d\,(a^2 + am - b)\ \therefore\ a^2 + am - b = \dfrac{r}{d}$ .... (2.)

Also $ab + bm - c = a^3 + a^2m - ab$ .... (3.)

$ca + cm - d = a^2b + abm - b^2$ .... (4.)

$ad + dm = ca^2 + acm - bc$, $\therefore d = \dfrac{ca^2 + acm - bc}{a + m}$ ... (5.)

$\therefore ca + cm - d = c(a + m) - d = c(a + m) - \dfrac{c(a^2 + am - b)}{a + m}$

$= \dfrac{c(a + m)^2 - c\{a(a + m) - b\}}{a + m} = \dfrac{c(a+m)\ (a + m - a) + bc}{a + m}$

$= \dfrac{c(a + m)\ m + bc}{a + m} = a^2b + abm - b^2 = ab\ (a + m) - b^2$

$\therefore c = \dfrac{b(a + m)\{a(a + m) - b\}}{m(a + m) + b}$; and from equation (3)

$ab + bm - c = b(a + m) - \dfrac{b(a + m)\{a(a + m) - b\}}{m(a + m) + b} =$

$\dfrac{mb(a + m)^2 + b^2(a + m) - ab\ (a + m)^2 + b^2\ (a + m)}{m(a + m) + b} =$

$\dfrac{2b^2(a + m) + b(a + m)^2\ (m - a)}{m(a + m) + b} = a^2(a + m) - ab$

or $2b^2(a + m) + b\ (a + m)^2\ (m - a) = ma^2(a + m)^2$ $+ ab(a - m)\ (a + m) - ab^2$ or $\{2(a + m) + a\}\ b^2 +$ $\{(a + m)^2\ (m - a) - a(a + m)\ (a - m)\}\ b = ma^2(a + m)^2$ or $(3a + 2m)b^2 + \{(a + m)\ (m - a)\ (2a + m)\}\ b = ma^2(a + m)^2$ or $b^2 + \dfrac{(a + m)\ (2a + m)\ (m - a)}{3a + 2m}\ b = \dfrac{ma^2(a + m)^2}{3a + 2m}$, $\therefore b =$

$\dfrac{-(a + m)(2a + m)(m - a) + \sqrt{(a + m)^2\{(2a + m)^2(m - a)^2 + 4ma^2(3a + 2m)\}}}{2\ (3a + 2m)}$

$= \dfrac{-(a + m)(2a + m)(m - a) + \sqrt{(4a^4 + 8a^3m + 5a^2m + 2am^3 + m^4)(a + m)^2}}{2\ (3a + 2m)} =$

$\dfrac{-(a + m)(2a + m)(m - a) + (a + m)^2\sqrt{4a^2 + m^2}}{2\ (3a + 2m)}$, since $m - a = -(a - m)$

$\therefore b = \dfrac{(a + m)\ (2a + m)\ (a - m) + (a + m)^2\sqrt{4a^2 + m^2}}{2\ (3a + 2m)}$. From (2) $a(a + m)$

$- b = \dfrac{2a(3a + 2m)\ (a + m) - (a + m)\ (2a + m)\ (a - m) - (a + m)^2\sqrt{4a^2 + m^2}}{2\ (3a + 2m)}$

It is evident that $2a(3a+2m)(a+m) = 6a^3 + 10a^2m + 4am^2$ and $(a+m)(2a+m)(a-m) = 2a^3 + a^2m - 2am^2 - m^3$ $\therefore$ $2a(3a+2m)(a+m) - (a+m)(2a+m)(a-m) = 4a^3 + 9a^2m + 6am^2 + m^3 = (a^2 + 2ma + m^2)(4a+m) = (a+m)^2(4a+m)$, therefore we find $a^2 + am - b = \dfrac{(a+m)^2(4a+m) - (a+m)^2\sqrt{4a^2+m^2}}{2(3a+2m)}$. From (1), $x = \dfrac{a+m}{2} + \sqrt{\dfrac{(a+m)^2}{4} - \dfrac{r}{d}}$ $\therefore$ when $r = \text{max.}$, $\dfrac{(a+m)^2}{4} = \dfrac{(a+m)^2(4a+m) - (a+m)^2\sqrt{4a^2+m^2}}{2(3a+2m)}$ $\therefore 1 = \dfrac{2(4a+m) - 2\sqrt{4a^2+m^2}}{3a+2m}$ $\therefore 5a = 2\sqrt{4a^2+m^2}$ or $25a^2 = 16a^2 + 4m^2$ or $a = \dfrac{2m}{3}$, and $x = \dfrac{a+m}{2} = \dfrac{\dfrac{2m}{3} + m}{2} = \dfrac{5m}{6}$.

If $m = 1$, then $x = \dfrac{5}{6}$.

PROB. (15.) TO FIND SUCH A VALUE OF x AS MAY MAKE $mx^4 - x^6 = \text{MAX.} = r$.

Let $y = x^2$ $\therefore$ $my^2 - y^3 = \text{max.}$ By Prob. chap. 2nd, we must have $y = \dfrac{2m}{3}$ or $x^2 = \dfrac{2m}{3}$ $\therefore x = \sqrt{\dfrac{2m}{3}}$. If $m = 1$, then $x = \sqrt{\dfrac{2}{3}}$.

PROB. (16.) TO FIND SUCH A VALUE OF x AS MAY MAKE $mx^3 - x^6 =$ MAX.

Let $y = x^3$ $\therefore$ $my - y^2 =$ max. then by Prob. chap. 1st, we must have $y = \frac{m}{2}$ or $x^3 = \frac{m}{2}$ $\therefore x = \sqrt[3]{\frac{m}{2}}$. If $m = 1$ then $x = \frac{1}{\sqrt[3]{2}}$.

PROB. (17.) TO FIND SUCH A VALUE OF x AS MAY MAKE $mx^2 - x^6 =$ MAX.

Let $x^2 = y$ $\therefore$ $my - y^3 =$ max. then by Prob. chap. 2nd, we must have $y = \sqrt{\frac{m}{3}}$ or $x^2 = \sqrt{\frac{m}{3}}$ $\therefore x = \sqrt[4]{\frac{m}{3}}$. If $m = 1$, then $x = \frac{1}{\sqrt[4]{3}}$.

PROB. (18.) TO FIND SUCH A VALUE OF x AS MAY MAKE $mx - x^6 =$ MAX. $= r$.

Since we have the equation $mx - x^4 = r$ or $x^6 - mx + r + 0$, let the product of four values of this equation $= x^4 + ax^3 + bx^2 + cx + d$,

$$\therefore x^4+ax^3+bx^2+cx+d\,)\; x^6-mx+r=0\; (\,x^2-ax+a^2-b=0 \ldots (1.)$$

$$\underline{x^6 + ax^5 + bx^4 + cx^3 + dx^2}$$

$$- ax^5 - bx^4 - cx^3 - dx^2 - mx$$

$$\underline{- ax^5 - a^2x^4 - abx^3 - acx^2 - adx}$$

$$(a^2-b)x^4 + (ab-c)x^3 + (ac-d)x^2 + (ad-m)x$$

$$\underline{(a^2-b)x^4 + (a^3-ab)x^3 + (a^2b-b^2)x^2 + (a^2c-bc)x}$$

$$+ r$$

$$\underline{+ d(a^2 - b)}$$

$\therefore d(a^2 - b) = r$ or $a^2 - b = \frac{r}{d}$ .... (2.)

Also $ab - c = a^3 - ab$ .... (3.)

$ac - d = a^2b - b^2$ .... (4.)

$ad - m = ca^2 - bc$ .... (5.)

Equation (5) gives $d = \dfrac{ca^2 - bc + m}{a}$ $\therefore$ $ac - d = ac - \dfrac{ca^2 - bc + m}{a} = \dfrac{bc - m}{a} = a^2b - b^2$ $\therefore$ $c = \dfrac{a^3b - ab^2 + m}{b}$ $\therefore$ $ab - c = ab - \dfrac{a^3b - ab^2 + m}{b} = \dfrac{2ab^2 - a^3b - m}{b} = a^3 - ab$, $\therefore$ $2ab^2 - a^3b - m = a^3b - ab^2$ $\therefore$ $3ab^2 - 2a^3b = m$ $\therefore$ $b^2 - \dfrac{2a^2}{3}b = \dfrac{m}{3a}$ and $\therefore$ $b = \dfrac{a^3 + \sqrt{a^6 + 3am}}{3a}$ $\therefore$ $a^2 - b = a^2 - \dfrac{a^3 + \sqrt{a^6 + 3am}}{3a} = \dfrac{2a^3 - \sqrt{a^6 + 3am}}{3a}$. From (1) we find, $x = \dfrac{a}{2} + \sqrt{\dfrac{a^2}{4} - \dfrac{r}{d}}$, $\therefore$ when $r =$ max. then $\dfrac{a^2}{4} = \dfrac{r}{d} = a^2 - b = \dfrac{2a^3 - \sqrt{a^6 + 3am}}{3a}$ and therefore $3a^3 = 8a^3 - 4\sqrt{a^6 + 3am}$ $\therefore$ $25a^6 = 16a^6 + 48am$, $\therefore$ $a^5 = \dfrac{16m}{3}$, and $x = \dfrac{a}{2}$, $\therefore$ $x^5 = \dfrac{a^5}{32} = \dfrac{16m}{3.32} = \dfrac{m}{6}$ $\therefore$ $x = \sqrt[5]{\dfrac{m}{6}}$. If $m=1$, then $x = \dfrac{1}{\sqrt[5]{6}}$.

It may be remarked here that all the problems of the two last sections of this chapter may be solved without impossible roots, in the manner laid down in preceding chapters.

CHAPTER IV.

PROBLEMS OF MAXIMA AND MINIMA IN WHICH TWO OR MORE VARIABLE QUANTITIES ARE USED.

If there are two variable quantities, find the value of each in terms of the other, according to the conditions of maximum or minimum, and it is evident that by this means we will find two equations by the comparison of which the values of the two variable unknown quantities will be found in terms of known constant quantities. If there be three variable quantities, find the value of each in terms of the other two, and thus make three equations, by means of which the values of the three unknown quantities will be determined. The same method may be adopted when there are four or more variable quantities.

The reason of this rule is obvious. When the being of maximum or minimum of any function depends on the values of two variables, for instance, then it is evident that the value of a single variable in terms of the other, found on the function being a maximum or minimum, will, itself, be a variable quantity, since the other variable is not yet determined; and consequently there will be infinite maxima or minima of the function proposed. Now, in order to find the required maximum or minimum out of these, we must solve the function with regard to both for their maximum or minimum values, then compare these two values, and thus determine them. The same reasoning may be applied in the case of functions of three or more variables

PROB. (1.) TO INSCRIBE THE GREATEST PARALLELOPIPEDON WITHIN A GIVEN ELLIPSOID.

Let $2x, 2y, 2z$ be the edges, $2a, 2b, 2c$ the principal diameters of the ellipsoid $\therefore$ the contents of the parallelopipedon $= 8xyz = u$, and by what is shown in the Introduction we find the equation of the ellipsoid to be

$$\frac{z^2}{c^2} + \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\therefore \; z^2 = c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$ and $\therefore$ square of $8xyz = 64x^2y^2z^2 = 64x^2y^2 \times c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 64c^2 \times \left(x^2y^2 - \frac{x^4y^2}{a^2} - \frac{x^2y^4}{b^2}\right) = \frac{64c^2}{a^2b^2}(a^2b^2x^2y^2 - b^2x^4y^2 - a^2x^2y^4) =$ max. and $\therefore a^2b^2x^2y^2 - b^2x^4y^2 - a^2x^2y^4 =$ max. First let x be considered as constant and y as variable $\therefore a^2b^2x^2y^2 - b^2x^4y^2 - a^2x^2y^4 = a^2x^2\left(\frac{a^2b^2y^2 - b^2x^2y^2}{a^2} - y^4\right) =$ max. $\therefore$ $\frac{a^2b^2y^2 - b^2x^2y^2}{a^2} - y^4 = \text{max.} = r \therefore y^4 - \frac{a^2b^2 - b^2x^2}{a^2}y^2 = -r \therefore y^2 = \frac{a^2b^2 - b^2x^2}{2a^2} \pm \sqrt{\frac{(a^2b^2 - b^2x^2)^2}{4a^4} - r}$, $\therefore$ when $r =$ max. we must have $\frac{(a^2b^2 - b^2x^2)^2}{4a^2} = r$, and $\therefore y^2 =$

$$\frac{a^2b^2 - b^2x^2}{2a^2} \quad \text{.........} \quad (1.)$$

Now let y be considered as constant and x as variable, $\therefore$ $a^2b^2x^2y^2 - a^2x^2y^4 - b^2y^2x^4 = b^2y^2\left(\frac{a^2b^2 - a^2y^2}{b^2}x^2 - x^4\right) =$ max. $\therefore \frac{a^2b^2 - a^2y^2}{b^2}x^2 - x^4 = \text{max.} = r, \therefore x^4 - \frac{a^2b^2 - a^2y^2}{b^2}$

$$-r, \therefore x^2 = \frac{a^2b^2 - a^2y^2}{2b^2} \pm \sqrt{\frac{(a^2b^2 - a^2y^2)^2}{4b^2}}$$

$\frac{(a^2b^2 - a^2y^2)^2}{4b^2} = r$, when $r = \text{max.} \therefore x^2 = \frac{a^2b^2 - a^2y^2}{2b^2} \ldots$ (2.)

$\therefore y^2 = \frac{a^2b^2 - 2b^2x^2}{a^2}$. Comparing this equation with equation (1) we find $\frac{a^2b^2 - b^2x^2}{2a^2} = \frac{a^2b^2 - 2b^2x^2}{a^2} = \frac{2a^2b^2 - 4b^2x^2}{2a^2}$

$\therefore 3b^2x^2 = a^2b^2$, $\therefore x^2 = \frac{a^2}{3}$, $\therefore x = \frac{a}{\sqrt{3}}$ and $\therefore$ equation

(1) gives $y^2 = \frac{a^2b^2 - \frac{a^2b^2}{3}}{2a^2} = \frac{2a^2b^2}{6a^2} = \frac{b^2}{3} \therefore y = \frac{b}{\sqrt{3}}$ and

$z^2 = c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = c^2\left(1 - \frac{1}{3} - \frac{1}{3}\right) = \frac{c^2}{3}$, $\therefore z = \frac{c}{\sqrt{3}}$. If v = volume of ellipsoid, $v = \frac{4}{3}\,pabc$ where $p = 3.14$ &c. $\therefore u = \frac{2v}{p\sqrt{3}}$ or $u : v :: 2 : p\sqrt{3}$.

PROB. (2.) GIVEN THE SUM OF THE LENGTHS OF THE THREE AXES OF AN ELLIPSOID, FIND THE LENGTH OF EACH, THAT THE VOLUME OF THE ELLIPSOID MAY BE A MAXIMUM.

Let x, y, z be the three axes and s their sum $\therefore x + y + z = s \therefore z = s - x - y$ and the volume of the ellipsoid $= \frac{4}{3}pxyz = \frac{4}{3}pxy\,(s - x - y) = \frac{4}{3}p\,(sxy - x^2y - xy^2) = \text{max.}$ $\therefore sxy - x^2y - xy^2 = \text{max.}$ First let x = a constant quantity, $\therefore x\,(sy - xy - y^2) = x\,\{(s - x)\,y - y^2\} = \text{max.}$ and $\therefore (s - x)\,y - y^2 = \text{max.} = r \therefore y^2 - (s - x)\,y = -r$, and $\therefore y = \frac{s - x}{2} \pm \sqrt{\frac{(s - x)^2}{4} - r}$. Here it is evident that when r = max. we must have $\frac{(s - x)^2}{4} = r$, $\therefore y = \frac{s - x}{2}$ .... (1.)

Now let y = a constant quantity, $\therefore sxy - x^2y - xy^2 = y(sx - x^2 - xy) = y\{(s-y)x - x^2\}$ = max. $\therefore (s-y)x - x^2$ = max. $= r, \therefore x^2 - (s-y)x = -r, \therefore x = \frac{s-y}{2} \pm \sqrt{\frac{(s-y)^2}{4} - r}$, and $\therefore \frac{(s-y)^2}{4} = r$, when r = max. $\therefore x = \frac{s-y}{2}, \therefore y = s - 2x$. Equation (1) gives $y = \frac{s-x}{2}, \therefore s - 2x = \frac{s-x}{2} = 2s - 4x = s - x \therefore 3x = s, \therefore x = \frac{s}{3}$ and $y = s - 2x = s - \frac{2s}{3} = \frac{s}{3}$ and $z = s - x - y = s - \frac{s}{3} - \frac{s}{3} = \frac{s}{3}$, and hence it appears that the axes of the ellipsoid required, when a maximum, must be equal to each other; that is to say, the ellipsoid required must be a sphere.

The same may easily be solved without impossible roots.

PROB. (3.) TO FIND THE VALUES OF x AND y, WHEN $x^3 + y^3 - 3axy$ = MAX. OR MIN.

In the first case let x = a constant quantity and find the value of y which will make $x^3 + y^3 - 3axy$ = max. $= p \therefore y^3 - 3axy = p - x^3 = r, \therefore y^3 - 3axy - r = 0$. Now let b = one of the negative roots of this equation, and $\therefore y + b$ must exactly divide it.

$$
\begin{array}{l}
y + b \,)\, y^3 - 3axy - r = 0 \,(\, y^2 - by + b^2 - 3ax = 0, \text{ (A.)} \\
\quad \underline{y^3 + by^2} \\
\qquad - by^2 - 3axy \\
\qquad \underline{- by^2 - b^2y} \\
\qquad\qquad (b^2 - 3ax)\, y - r \\
\qquad\qquad \underline{(b^2 - 3ax)\, y + b(b^2 - 3ax)} \quad \therefore b(b^2 - 3ax) \\
\qquad\qquad\qquad\qquad [= -r \therefore b^2 - 3ax = -\frac{r}{b}
\end{array}
$$

$\therefore 3ax - b^2 = \frac{r}{b}$ and $3abx - b^3 = r$. Now equation (A) gives $y^2 - by = \frac{r}{b}$, $\therefore y = \frac{b}{2} \pm \sqrt{\frac{b^3 + 4r}{4b}} = \frac{b}{2} \pm \sqrt{\frac{-3b^3 + 4r + 4b^3}{4b}}$. Here it is evident that if $4r + 4b^3$ be a negative quantity, there shall never be a maximum or a minimum, and if $4r + 4b^3$ be positive, we shall have a minimum; for in this case we cannot suppose r so small or negatively so large as to make $4r + 4b^3$ less than $3b^3$ which is negative, $\therefore$ when $r =$ min. we must have $4r + 4b^3 = 3b^3$ or $12abx - 4b^3 + 4b^3 = 12abx = 3b^3$ or $b = 2\sqrt{ax}$ and $y = \frac{b}{2} = \sqrt{ax}$. When y is considered as constant, we can show, exactly in the manner above stated, that $x = \sqrt{ay}$, $\therefore x^2 = ay = a\sqrt{ax}$, $\therefore x^4 = a^3x$, $\therefore x^3 = a^3$, $\therefore x = a$, and $y = \sqrt{ax} = \sqrt{a^2} = a$.

The same solved without impossible roots.

In the equation $y^2 - by = \frac{r}{b}$, let $y = z + \frac{b}{2}$ and therefore $y^2 - by = z^2 + bz + \frac{b^2}{4} - bz - \frac{b^2}{2} = z^2 - \frac{b^2}{4} = \frac{r}{b}$ $\therefore r = bz^2 - \frac{b^3}{4}$. Here it is evident that r becomes a minimum by being negatively large (for $r = p - x^3$) and $\therefore$ when $r =$ min. we must have $z = 0$, $\therefore r = -\frac{b^3}{4} = 3abx - b^3$, $\therefore -b^3 = 12abx - 4b^3$ $\therefore 3b^3 = 12abx$ $\therefore b = 2\sqrt{ax}$ as before.

PROB. (4.) TO FIND THE VALUES OF x AND y SUCH THAT $x^2y^3(a - x - y) =$ MAX.

First let x be considered as variable and y as constant, $\therefore x^3(a - x - y) = (a - y)x^3 - x^4 =$ max. or $Ax^3 - x^4 =$

max. where $a - y = A$. Now proceeding exactly as in Prob. (4), chapter 3rd, we find $x = \frac{3A}{4} = \frac{3(a-y)}{4}$ ...... (1)

Now let x be constant and y be variable, and dividing the given expression by x^3 we find $y^2(a - x - y) = (a - x)$ $y^2 - y^3 = By^2 - y^3 =$ max. where $a - x = B$. Now proceeding exactly as in Prob. (4), chapter 2nd, we find $y = \frac{2B}{3} = \frac{2}{3}(a - x) = \frac{2}{3}\left\{a - \frac{3}{4}(a - y)\right\}$ from equation (1)

$\therefore y = \frac{2}{3}\left(\frac{a}{4} + \frac{3y}{4}\right) = \frac{2a}{12} + \frac{6y}{12} = \frac{a}{6} + \frac{y}{2}$, $\therefore y = \frac{a}{3}$ and

$\therefore x = \frac{3}{4}(a - y) = \frac{3}{4}\left(a - \frac{a}{3}\right) = \frac{3}{4} \times \frac{2a}{3} = \frac{a}{2}$.

The same may be solved without impossible roots as problems in the preceding chapters.

PROB. (5.) GIVEN THE PERIMETER OF A TRIANGLE ABC, SHOW THAT ITS AREA IS THE GREATEST, WHEN IT IS EQUILATERAL.

Let $2p =$ the perimeter of the triangle required, $AC = x$, $AB = y$ and $\therefore$ $CB = 2p - x - y$, and consequently, by what is shown in the introductory chapter, the area of this triangle $= \sqrt{p(p-x)\ (p-y)\ (x+y-p)} =$ max. and $\therefore$ $p(p-x)\ (p-y)\ (x+y-p) =$ max. Now when x is constant, we find $(p-y)\ (x+y-p) =$ max. $\therefore -p^2 + px + 2py - xy - y^2 =$ max. and since x and p are constants, we find $2py - xy - y^2 =$ max. $= r$, $\therefore y^2 + xy - 2py = -r$. $\therefore$ $y^2 - (2p - x)\ y = -r$, and solving this quadratic we find $y = \frac{2p - x}{2} \pm \sqrt{\frac{(2p-x)^2 - 4r}{4}}$ and $\because$ when $4r$ or $r =$ max. we must have $(2p - x)^2 = 4r$, and $\therefore y = \frac{2p - x}{2}$, (A.)

Now suppose y to be constant $\therefore (p-x)(x+y-p) =$ max. or $-p^2 + 2px + py - x^2 - xy =$ max. and since y and p are constants, we find $2px - x^2 - xy =$ max. $= r \therefore x^2 - (2p-y)x = -r$. Solving this quadratic we find $x = \frac{2p-y}{2} \pm \sqrt{\frac{(2p-y)^2 - 4r}{4}}$, and here it is evident that when r or $4r =$ max. then $(2p-y)^2 = 4r$, $\therefore x = \frac{2p-y}{2}$ .... (B.)

Comparing equations (A) and (B) we find $y = \frac{2p-x}{2} = \frac{2p - \frac{2p-y}{2}}{2} = \frac{2p+y}{4}$, $\therefore 4y = 2p + y$, $\therefore y = \frac{2}{3}p$, $\therefore x = \frac{2p-y}{2} = \frac{2p - \frac{2p}{3}}{2} = \frac{2}{3}p$, and the third side $= 2p - x - y = 2p - \frac{2}{3}p - \frac{2}{3}p = \frac{2}{3}p$, and hence it appears that the triangle required must be equilateral.

The same may be solved without impossible roots, as problems in the preceding chapters.

PROB. (6.) GIVEN THE SURFACE OF A RECTANGULAR PARALLELOPIPEDON, FIND WHEN THE CONTENT IS A MAXIMUM.

Let x, y, and $z =$ length, breadth, and thickness of the parallelopipedon and $2a =$ its surface. Now it is evident that the whole surface given must be $= 2xy + 2xz + 2yz = 2a$ or $xy + xz + yz = a$, and $\therefore z = \frac{a - xy}{x+y}$ and the content $= \frac{xy \times (a - xy)}{x+y} =$ max. When $x =$ constant and $y =$ variable $\therefore y \times \frac{a - xy}{x+y} =$ max. .... (A.)

and when y = constant and x = variable then $x \times \dfrac{a - xy}{x + y}$ = max. .... (B.)

Equation (A) gives $\dfrac{ay - xy^2}{x + y} = \text{max.} = r, \therefore ay - xy^2 = rx + ry, \therefore y^2 - \dfrac{a - r}{x} y = -r, \therefore y = \dfrac{a - r}{2x} \pm \sqrt{\dfrac{(a-r)^2 - 4x^2 r}{4x^2}}$ and hence it is evident that as r is greater, so $(a - r)^2$ becomes less, and $4x^2 r$ greater, and $\therefore$ when r = max. we must have $(a - r)^2 = 4x^2 r$ or $a^2 - 2ar + r^2 = 4x^2 r$, and $\therefore r^2 - 2(a + 2x^2) r = - a^2, \therefore r = a + 2x^2 \pm 2x\sqrt{a + x^2}, \therefore y =$

$\overline{+ x^2}$ ...... (C). When y = cons

then from equation (B), $\dfrac{ax - yx^2}{x + y}$ = max. = r; and exactly as above we find $x = - y - \sqrt{a + y^2}$ .... (D.)

From equation (C), $y + x = -\sqrt{a + x^2}, \therefore y^2 + 2xy + x^2 = a + x^2, \therefore x = \dfrac{a - y^2}{2y}$, and from equation (D) we find $\dfrac{a - y^2}{2y} = - y - \sqrt{a + y^2}, \therefore \dfrac{a + y^2}{2y} = - \sqrt{a + y^2}$, and $\therefore$ $\dfrac{a^2 + 2ay^2 + y^4}{4y^2} = a + y^2$ or $\dfrac{a + y^2}{4y^2} = 1, \therefore a + y^2 = 4y^2 \therefore$

$$y^2 = \frac{a}{3}, \therefore y = \sqrt{\frac{a}{3}} \text{ and } x = \frac{a - y^2}{2y} = \frac{a - \frac{a}{3}}{2\sqrt{\frac{a}{3}}} = \sqrt{\frac{a}{3}}$$

$$\text{and } z = \frac{a - xy}{x + y} = \frac{a - \frac{a}{3}}{2\sqrt{\frac{a}{3}}} = \sqrt{\frac{a}{3}}, \therefore x = y = z = \sqrt{\frac{a}{3}}$$

and $\therefore xy + xy + yz = x^2 + x^2 + x^2 = 3x^2 = 3 \times \dfrac{a}{3} = a$; hence it appears that the required parallelopipedon is a cube.

The same may easily be solved without impossible roots.

PROB. (7.) . INSCRIBE THE GREATEST TRIANGLE ABC WITHIN A GIVEN CIRCLE. (Fig. 64.)

Let $R =$ radius of the given circle, a, b, c, the unknown sides of the triangle required, $n = \angle B$, $m = \angle C$. $\therefore$ area of the triangle $= \dfrac{BC \times AD}{2} = \dfrac{BC.AB.AC}{4R}$, because by 6 B. Euc. we have $2R \times AD = AB.AC$. $\therefore 2Rc \sin. n = cb$ $\therefore b = 2R \sin. n$ $\therefore c = b\dfrac{\sin. m}{\sin. n} = 2R \sin. m$ and $a = 2R \sin. A = 2R \sin. (m + n)$ $\therefore$ area required $= \dfrac{abc}{4R} = 2R^2 \sin. n \sin. m \times \sin. (n + m) =$ max. $\therefore \sin. n \sin. m \sin. (m + n) =$ max. Now let $\sin. n = x$ $\therefore \cos n = \sqrt{1 - x^2}$, $\sin. m = y$ $\therefore \cos m = \sqrt{1 - y^2}$ $\therefore xy\,(x\sqrt{1 - y^2} + y\sqrt{1 - x^2}) =$ max. First suppose that $x =$ constant, $\therefore y(x\sqrt{1 - y^2} + y\sqrt{1 - x^2})$ or $\sqrt{y^2 - y^4} + \dfrac{y^2\sqrt{1 - x^2}}{x} =$ max. Let $\dfrac{\sqrt{1 - x^2}}{x} = A$ $\therefore$ $\sqrt{y^2 - y^4} + Ay^2 =$ max. $= r$ $\therefore y^2 - y^4 = r^2 - 2Ary^2 + A^2y^4$ $\therefore y^4 - \dfrac{2Ar + 1}{A^2 + 1}y^2 = -\dfrac{r^2}{A^2 + 1}$ $\therefore y^2 = \dfrac{2Ar + 1}{2(A^2 + 1)} \pm \sqrt{\dfrac{4r(A - r) + 1}{4(A^2 + 1)^2}}$ and here it is evident that r cannot be taken so great, as becoming greater than A may make $4r(A - r)$ = a negative quantity greater than one, for in this case the root becomes impossible, and therefore when $r =$ max. we must have $4r(A - r) = -1$ $\therefore r^2 - Ar = \dfrac{1}{4}$ $\therefore r = \dfrac{A + \sqrt{A^2 + 1}}{2}$ and $y^2 = \dfrac{2Ar + 1}{2\,(A^2 + 1)} = \dfrac{A^2 + 1 + A\sqrt{A^2 + 1}}{2(A^2 + 1)}$ but $A^2 = \dfrac{1 - x^2}{x^2}$ $\therefore y^2 = \dfrac{1 + \sqrt{1 - x^2}}{2}$ or $\sin.^2 m =$

$$\frac{1 + \cos n}{2} \quad \text{...} \quad (A.)$$

In exactly the same manner as shown above we may find,

$$\sin.^2 n = \frac{1 + \cos m}{2} \quad \text{.......................} \quad (B.)$$

when $y =$ constant and $\frac{\sqrt{1 - y^2}}{y}$ is supposed $= A$. From (B) we find

$1 - \sin.^2 n = \cos^2 n = 1 - \frac{1 + \cos m}{2} = \frac{1 - \cos m}{2}$; $\therefore$ but equation (A) gives $\cos^2 n = (1 - 2\sin.^2 m)^2 = \frac{1 - \cos m}{2}$ or $\cos^2 2m = \frac{1 - \cos m}{2}$ $\therefore \cos m = 1 - 2\cos^2 2m = -(2\cos^2 2m - 1) = -\cos 4m \therefore \cos m = -\cos 4m$. Hence it appears that m is such an angle that its cosine is equal to the negative cosine of its quadruple $\therefore m = 60°$ Now from equation (B) $\sin.^2 m = \frac{1 + \cos 60°}{2} = \frac{1 + \frac{1}{2}}{2} = \frac{3}{4} \therefore \sin. n = \frac{\sqrt{3}}{2} \therefore n$ also $= 60° \therefore$ the third angle $= A = 180 - 60° - 60° = 60° \therefore$ the triangle required is equiangular and equilateral. One of its sides $= a = 2R \sin. A = 2R \times \frac{\sqrt{3}}{2} = R\sqrt{3} = b = c$.

The same may easily be solved without impossible roots.

PROB. (8.) TO FIND THAT POINT WITHIN A GIVEN TRIANGLE, FROM WHICH IF LINES BE DRAWN TO THE ANGULAR POINTS, THE SUM OF THEIR SQUARES SHALL BE A MINIMUM. (Fig. 65.)

Let ABC be the given triangle, and let $BD = a$, $AC = b$, $AD = c$, $AE = x$, $EG = y$ where G is the point required, $\therefore DE = x - c$, $FB = a - y$, and $EC = b - x$, and therefore $AG^2 + CG^2 + GB^2 = x^2 + y^2 + (b - x)^2 + y^2 + (x - c)^2 + (a - y)^2 = 3x^2 + 3y^2 - 2(b + c)x - 2ay + a^2 + b^2 + c^2 = \max. \therefore x^2 + y^2 - \frac{2(b + c)}{3}x - \frac{2a}{3}y + \frac{a^2 + b^2 + c^2}{3} =$

max. $= r$. First let $y =$ constant and x variable, $x^2 - \frac{2(b+c)}{3}x = r - \frac{a^2+b^2+c^2}{3} - y^2 + \frac{2a}{3}y$ and $\therefore x =$
$\frac{b+c}{3} \pm \sqrt{r - \frac{a^2+b^2+c^2}{3} + \frac{b^2+2bc+c^2}{9} - y^2 + \frac{2a}{3}y}$
$= \frac{b+c}{3} \pm \sqrt{r + \frac{2bc - 3a^2 - 2b^2 - 2c^2}{9} - y^2 + \frac{2a}{3}y}$.
By inspecting the diagram it is manifest that $2b^2 > 2bc$ $\therefore 2bc - 2b^2 =$ a negative quantity $= -n$, $\therefore$ we find

$$x = \frac{b+c}{3} \pm \sqrt{r - \frac{n+3a^2+2c^2}{9} - y^2 + \frac{2a}{3}y}$$

$$\frac{b+c}{3} \pm \sqrt{r - \frac{n}{9} - \frac{2c^2}{9} - \frac{3a^2}{9} - y^2 + \frac{2a}{3}y}.$$

Now we say that $\frac{3a^2}{9} + y^2$ is $> \frac{2a}{3}y$; if it is not so, 1st, let $\frac{3a^2}{9} + y^2 = \frac{2a}{3}y$, $\therefore y^2 - \frac{2a}{3}y = -\frac{3a^2}{9}$, $\therefore y = \frac{a \pm \sqrt{-2a^2}}{3} =$ an imaginary quantity; 2nd, let $\frac{3a^2}{9} + y^2 < \frac{2a}{3}y$, and $\therefore$ let $\frac{3a^2}{9} + y^2 + P = \frac{2a}{3}y$, $\therefore y^2 - \frac{2a}{3}y = -\frac{3a^2}{9} - P$, $\therefore y = \frac{a \pm \sqrt{-2a^2 - 9P}}{3} =$ an imaginary quantity. Hence $-\frac{3a^2}{9} - y^2 + \frac{2a}{3}y =$ a negative quantity $= -m$, suppose; $\therefore x = \frac{b+c}{3} \pm \sqrt{r - \frac{n}{9} - \frac{2c^2}{9} - m}$, and $\therefore$ when $r =$ min. then $r = \frac{n}{9} + \frac{2c^2}{9} + m$ and $x = \frac{b+c}{3}$ ...... (1). When $x =$ a constant, then from the original equation we find
$y^2 - \frac{2a}{3}y = r - \frac{a^2+b^2+c^2}{3} - x^2 + \frac{2(b+c)}{3}x$ and as above it may be shown that when $r =$ min. we must have $r = \frac{2a^2+3b^2+3c^2}{9} + x^2 - \frac{2(b+c)}{3}x$, $\therefore y = \frac{a}{3}$ ... (2.)

The same may easily be solved without impossible roots.

PROB. (9.) TO FIND A POINT WITHIN A TRIANGULAR PYRAMID, FROM WHICH IF LINES BE DRAWN TO THE ANGULAR POINTS, THE SUM OF THEIR SQUARES IS THE LEAST POSSIBLE. (Fig. 66.)

Let $ACEB$ be the given pyramid, ABC its given base, $EG = a =$ perpendicular drawn from the vertex to the base of the pyramid. Let K be the point required and the perpendicular drawn from this point to the base $= KH$ and HD $=$ perpendicular from H to $AC = y$, and $AD = x$, $GF =$ perpendicular from G to $AC = b$, and $AF = c$. Let Hn be drawn parallel to DF, $\therefore$ $Hn = DF = c - x$ and $Gn = GF - HD = b - y$, $\therefore$ $HG^2 = (c - x)^2 + (b - y)^2$. Also let $AC = d$, $\therefore DC = d - x$. Draw Kl parallel to HG and $\therefore$ $HG^2 = Kl^2$. Join K, B and B, H, and now it is evident the $\angle$ $KHB =$ a right angle. By a process exactly similar to that used in the foregoing proposition, it may be shown that $HB^2 = (d - x)^2 + (e - y)^2$ where e is the altitude of the triangular base of the pyramid. Let $KH = z$ $\therefore$ $KB^2 =$
$(d - x)^2 + (e - y)^2 + z^2$ .... (1.)

It is manifest that $AH^2 + KH^2 = AD^2 + HD^2 + KH^2 =$
$x^2 + y^2 + z^2$ .... (A.)

$CK^2 = CH^2 + KH^2 = CD^2 + HD^2 + KH^2 = (d - x)^2 +$
$y^2 + z^2 = d^2 - 2dx + x^2 + y^2 + z^2$ .... (B.)

$KE^2 = Kl^2 + lE^2 = Kl^2 + (EG - KH)^2$

$= (c - x)^2 + (b - y)^2 + (a - z)^2$

$= c^2 - 2cx + b^2 - 2by + a^2 - 2az + x^2 + y^2 + z^2$

$= a^2 + b^2 + c^2 - 2cx - 2by - 2az + x^2 + y^2 + z^2$...(C.)

From equation (1) we find

$KB^2 = d^2 + e^2 - 2dx - 2ey + x^2 + y^2 + z^2$ .... (D.)

Adding together these four equations we find

$AK^2 + BK^2 + CK^2 + EK^2 =$

$4x^2 + 4y^2 + 4z^2 + a^2 + b^2 + c^2 + 2d^2 + e^2 - 2cx - 2by -$
$2az - 4dx - 2ey =$

$$4\left(x^2 + y^2 + z^2 - \frac{c + 2d}{2} x - \frac{b + e}{2} y - \frac{a}{2} z + \frac{a^2 + b^2 + c^2 + 2d^2 + e^2}{4}\right).$$

Now as $\frac{a^2+b^2+c^2+2d^2+e^2}{4}$ = a constant $\therefore$ when z, or y, or x, are supposed to be constants respectively, we shall have severally the following three equations, the second members of which must be such as to become negative when the orignal minimum quantities are taken very small, for these second members are nothing more than the difference of the minimum quantities supposed and constant quantities taken to the other sides of the equations.

$x^2 - \frac{c + 2d}{2} x = \text{min.} = r$ when y and z are constants,

$y^2 - \frac{b + e}{2} y = \text{min.} = r$ when x and z are constants,

$z^2 - \frac{a}{2} z = \text{min.} = r$ when x and y are constants, and from these equations we find

$x = \frac{c+2d}{4} \pm \sqrt{\frac{(c+2d)^2}{16} + r}$, $y = \frac{b+e}{4} \pm \sqrt{\frac{(b+e)^2}{16} + r}$

and $z = \frac{a}{4} \pm \sqrt{\frac{a^2}{16} + r}$, and here it is evident that r cannot be taken so small or negatively so large, as to make the roots impossible, and therefore when r = min. we must have $\frac{(c + 2d)^2}{16} + r = 0$, $\frac{(b + c)^2}{16} + r = 0$, and $\frac{a^2}{16} + r = 0$, and $\therefore x = \frac{c + 2d}{4}$, $y = \frac{b + c}{4}$ and $z = \frac{a}{4}$.

The same may easily be solved without impossible roots.

*** The symbol r is used in three different senses.—Ed.

PROB. (10.) TO FIND VALUES OF x AND y SUCH AS WILL MAKE $(x + 1)(y + 1)(z + 1) = \text{MAX.}$ ...... (1) WHERE $a^x b^y c^z = A$ ...... (2).

Taking logarithms of the equation (2) we find $x \log a + y \log b + z \log c = \log A$ and let $\log a = p$, $\log b = m$, $\log c = n$ and $\log A = q$ .... (3)
$\therefore px + my + nz = q$, $\therefore z = \dfrac{q - px - my}{n}$ $\therefore z + 1 = \dfrac{q + n - px - my}{n}$; substituting this value of $z + 1$ in (1) we find

$(x + 1)(y + 1)\dfrac{(q + n - px - my)}{n} = \text{max.}$ or $(x + 1)(y + 1)(q + n - px - my) = \text{max.}$ Now when $x + 1 =$ constant, we have $(y + 1)(q + n - px - my) = (q + n)y - pxy - my^2 - my - px + q + n = \text{max.}$ $\therefore$
$(q + n - m - px)y - my^2 - px + q + n = \left\{\dfrac{(q+n-m-px)}{m} y - y^2 - \dfrac{px - q - n}{m}\right\} m = \text{max.}$ $\therefore \dfrac{q + n - m - px}{m}$ $- y^2 - \dfrac{px - q - n}{m} = \text{max.}$ Now as $\dfrac{px - q - n}{m} =$ constant, we have $\dfrac{q + n - m - px}{m} y - y^2 = \text{max.} = r$, $\therefore$
$y^2 - \dfrac{q + n - m - px}{m} y = - r$. Solving this quadratic we find $y = \dfrac{q + n - m - px}{2m} \pm \sqrt{\dfrac{(q + n - m - px)^2}{4m^2} - r}$ and here it is evident that when $r = \text{max.}$ we must have $\dfrac{(q + n - m - px)^2}{4m^2} = r$, $\therefore y = \dfrac{q + n - m - px}{2m}$ ....(4.)
Now let $y =$ constant $\therefore (x + 1)(q + n - px - my) = \text{max.}$ $\therefore (q + n - my)x - px^2 - my + q + n - px =$
$(q + n - p - my)x - px^2 + q + n - my = \left\{\dfrac{q + n - p - my}{p}\right.$

$x - x^2 + \left.\frac{q+n-my}{p}\right\} \times p = \text{max.} \therefore \frac{q+n-p-my}{p}$

$x - x^2 + \frac{q+n-my}{p} = \text{max.}$, and as $\frac{q+n-my}{p} =$

constant, we must have $\frac{q+n-p-my}{p} x - x^2 = \text{max.} = r$

and therefore $x^2 - \frac{q+n-p-my}{p} x = -r$; solving this

quadratic we find $x = \frac{q+n-p-my}{2p} \pm \sqrt{\frac{(q+n-p-my)^2}{4p^2} - r}$

$\therefore$ when $r = \text{max.}$ then $x = \frac{q+n-p-my}{2p}$ .... (5.)

$\therefore px = \frac{q+n-p-my}{2}$. Substituting this value of px

in (4) we find $y = \dfrac{q+n-m-\dfrac{q+n-p-my}{2}}{2m} =$

$\frac{q+n-2m+p+my}{4m} \therefore y = \frac{q+n+p-2m}{3m}$ .... (6.)

Substituting the values of q, n, p, m from equations (3) we find $y = \frac{\log A + \log c + \log a - 2\log b}{3\log b} = \frac{\log(Aac) - 2\log b}{3\log b}$

$\therefore y + 1 = \frac{\log(Aabc)}{3\log b}$ .... (7.)

Substituting the value of y from equation (6) in (5) we find

$x = \dfrac{q+n-p-\dfrac{q+n+p-2m}{3}}{2p} = \dfrac{2q+2n-4p+2m}{2 \times 3p}$

$= \frac{q+n+m-2p}{3p}$. Now substituting the values of $q, n, m,$ and p from (3) we find $x = \frac{\log A + \log c + \log b - 2\log a}{3\log a}$

$= \frac{\log(Abc) - 2\log a}{3\log a} \therefore x + 1 = \frac{\log(Aabc)}{3\log a}$ .... (8.)

Now from equation $x\log a + y\log b + z\log c = \log A$, we find $z\log c = \log A - x\log a - y\log b$

$$= \log A - \frac{\log(Abc) - 2\log a}{3} - \frac{\log(Aac) - 2\log b}{3}$$

$$= \frac{3\log A - \log A - \log(bc) + 2\log a - \log A - \log(ac) + 2\log b}{3}$$

$$= \frac{\log A - \log(bc) - \log(ac) + 2\log a + 2\log b}{3}$$

$$= \frac{\log A - \log b - \log c - \log a - \log c + 2\log a + 2\log b}{3}$$

$$= \frac{\log A + \log b + \log a - 2\log c}{3} = \frac{\log(Aab) - 2\log c}{3}$$

$\therefore z + 1 = \dfrac{\log(Aabc)}{3\log c}$ $\therefore$ we find $(x + 1)(y + 1)(z + 1)$

$$= \text{max.} = \frac{\{\log(Aabc)\}^3}{27\log a \log b \log c}.$$

The same may easily be solved without impossible roots.

PROB. (11.) TO INSCRIBE A TRIANGLE WITHIN A GIVEN CIRCLE SO THAT ITS PERIMETER MAY BE A MAXIMUM. (Fig. 67.)

Let ABC be the triangle required. The centre of the given circle is E, and ED, EF, EG perpendiculars let fall from the centre on the sides of the triangle. Let the $\angle\, AEC = 2\theta$ $\therefore$ each of the angles AED, and $CED = \theta$. Likewise $AEF = FEB = \phi$ and $\therefore$ the $\angle\, BEC = 360° - 2\theta - 2\phi$ and $\therefore$ $BEG = GEC = \dfrac{360 - 2\theta - 2\phi}{2} = 180 - (\theta + \phi)$ and $\sin. BEG = \sin. (\theta + \phi)$. Also let the radius of the given circle $= \dfrac{a}{2}$. Now it is evident that $AD = \dfrac{a}{2}\sin.\theta$ $\therefore$ $AC = 2AD = a\sin.\theta$, and in like manner $AB = a\sin.\phi$, and $BC = a\sin(\theta + \phi)$ $\therefore$ perimeter $= a\{\sin\theta + \sin\phi + \sin(\theta + \phi)\}$

$= \text{max.} \therefore \sin.\theta + \sin.\phi + \sin.(\theta + \phi) = \text{max.}$ Now let $\sin.\theta = \text{constant} \therefore \sin\phi + \sin.\theta\cos.\phi + \sin.\phi\cos.\theta = (1 + \cos.\theta)\sin.\phi + \sin.\theta\cos.\phi = \text{max.}$ Let $1 + \cos.\theta = n$, $\sin.\phi = x$, $\sin.\theta = c$, $\therefore \cos.\phi = \sqrt{1 - x^2} \therefore nx + c\sqrt{1 - x^2} = \text{max.} = r$, $\therefore c^2 - c^2x^2 = r^2 - 2nrx + n^2x^2 \therefore (c^2 + n^2)x^2 - 2nrx = c^2 - r^2$, $\therefore x^2 - \frac{2nr}{c^2 + n^2}x = \frac{c^2 - r^2}{c^2 + n^2}$; solving this quadratic we find $x = \frac{nr + c\sqrt{c^2 + n^2 - r^2}}{c^2 + n^2} \therefore$ when $r = \text{max.}$ we must have $c^2 + n^2 = r^2 \therefore r = \sqrt{c^2 + n^2}$, $\therefore x = \frac{nr}{c^2 + n^2} = \frac{n}{\sqrt{c^2 + n^2}} =$

$$\frac{1 + \cos.\theta}{\sqrt{1 + 2\cos.\theta + \cos.^2\theta + \sin.^2\theta}} = \frac{1 + \cos.\theta}{\sqrt{2(1 + \cos.\theta)}} =$$

$\sqrt{\frac{1 + \cos.\theta}{2}}$, $\therefore x = \sin.\phi = \sqrt{\frac{1 + \cos.\theta}{2}}$. In like manner when $\sin.\phi = \text{constant}$, we may easily find $\sin.\theta = \sqrt{\frac{1 + \cos.\phi}{2}}$. Now let $\sin.\theta = y \therefore \cos.\theta = \sqrt{1 - y^2}$ and we have supposed $\sin.\phi = x \therefore \cos.\phi = \sqrt{1 - x^2} \therefore x^2 = \frac{1 + \sqrt{1 - y^2}}{2}$ and $y^2 = \frac{1 + \sqrt{1 - x^2}}{2}$ and $\therefore 4x^4 - 4x^2 + 1 = 1 - y^2 \therefore y^2 = 4x^2(1 - x^2)$ .... (2.)

Also $4y^4 - 4y^2 + 1 = 1 - x^2$ $\therefore 4x^2 = 16y^2 - 16y^4$; substituting this value of $4x^2$ and $1 - x^2$ in equation (2) we find $y^2 = (16y^2 - 16y^4)(4y^4 - 4y^2 + 1) = -64y^8 + 128y^6 - 80y^4 + 16y^2$ and $y^6 - 2y^4 + \frac{5}{4}y^2 - \frac{15}{64} = 0$.

Now let $y^2 = z$, $\therefore z^3 - 2z^2 + \frac{5}{4}z - \frac{15}{64} = 0$. This equation is exactly divisible by $z - \frac{3}{4}$ as may appear by actual division $\therefore \frac{3}{4} = $ a value of $z = y^2 \therefore y = \sqrt{\frac{3}{2}}$ and

$x^2 = \frac{1 + \sqrt{1 - y^2}}{2} = \frac{1 + \frac{1}{2}}{2} = \frac{3}{4}$ and $x = \sqrt{\frac{3}{2}}$ or sin. θ $= \sqrt{\frac{3}{2}}$ and $\sin \phi = \sqrt{\frac{3}{2}} \therefore \theta = \phi = 60^\circ$ and hence it appears that the triangle required is equiangular, and the sides $= \frac{a\sqrt{3}}{2}$ each where $a =$ radius.

The same may easily be solved without impossible roots.

PROB. (12.) TO FIND SUCH VALUES OF x, y, z AS WILL MAKE $\frac{xyz}{(x+a)(x+y)(y+z)(z+e)} =$ MAX.

First let y and $z =$ constant quantities $\therefore \frac{x}{(x+a)(x+y)}$ $=$ max. $\therefore \frac{(x+a)(x+y)}{x} = \frac{x^2 + (a+y)x + ay}{x} =$ min. $= r$, $\therefore x^2 - (r - a - y)x = -ay$. Solving this quadratic we find $x = \frac{r - a - y}{2} \pm \sqrt{\frac{(r-a-y)^2}{4} - ay}$, and here it is evident that when $r =$ min. then $\frac{(r-a-y)^2}{4} = ay \therefore$ $\frac{r-a-y}{2} = \sqrt{ay}$, $\therefore x = \frac{r-a-y}{2} = \sqrt{ay}$ .... (1.)

Secondly when x and $z =$ constants we find likewise $y = \sqrt{xz}$ .... (2.)

Thirdly when x and $y =$ constants we find $z = \sqrt{ye}$... (3.)

From (1) and (2) we find $x^2 = ay$ and $x^2 = \frac{y^4}{z^2}$ and $ay =$ $\frac{y^4}{z^2}$, $\therefore a = \frac{y^3}{z^2} \therefore y^3 = az^2$ and from (3) we find $\frac{z^2}{e} = y$, $\therefore y^3 =$ $\frac{z^6}{e^3} = az^2$, $\therefore z^4 = ae^3$, $\therefore z = \sqrt[4]{ae^3}$ and $y = \frac{z^2}{e} = \frac{\sqrt{ae^3}}{e} =$

$\sqrt{ae} = \sqrt[4]{a^2e^2}$, $x = \sqrt{ay}$, $\therefore x^4 = a^2y^2 = a^2 \times ae = a^3e$, $\therefore x = \sqrt[4]{a^3e}$. Hence it appears that x, y and z are in geometrical progression and the common ratio is $\sqrt[4]{\frac{e}{a}}$.

The same may easily be solved without impossible roots.

PROB. (13.) IF THE CONTENT OF A RECTANGULAR PARALLELOPIPEDON BE GIVEN, FIND ITS FORM WHEN THE SURFACE IS A MINIMUM.

Let the content of the parallelopipedon $= a = xyz$ $\therefore$ $z = \frac{a}{xy}$; and it is evident that half its surface must be $= xy + xz + yz =$ min. or $xy + \frac{a}{y} + \frac{a}{x} = \frac{x^2y^2 + ax + ay}{xy}$ $=$ min. First let $y =$ constant $\therefore$ $\frac{x^2y^2 + ax + ay}{x} =$ $\frac{\left(x^2 + \frac{ax}{y^2} + \frac{a}{y}\right)}{x} =$ min. $\therefore$ $\frac{x^2 + \frac{ax}{y^2} + \frac{a}{y}}{x} =$ min. $= r$, $\therefore$

$$x^2 - \left(r - \frac{a}{y^2}\right)x = -\frac{a}{y}, \therefore x = \frac{r - \frac{a}{y^2}}{2} \pm \sqrt{\frac{\left(r - \frac{a}{y^2}\right)^2}{4} - \frac{a}{y}}.$$

$\therefore$ when $r =$ min. we must have $\frac{r - \frac{a}{y^2}}{2} = \sqrt{\frac{a}{u}}$ $\therefore x$ $\sqrt{\frac{a}{u}}$. Likewise when $x =$ constant and $r =$ min. we find, $y = \sqrt{\frac{a}{x}}$ and $z = \frac{a}{xy} = \frac{a}{\frac{a}{\sqrt{xy}}} = \sqrt{xy}$; therefore $x^2 = \frac{a}{y}$ and $y^2 = \frac{a}{x}$ $\therefore$ $y^4 = \frac{a^2}{x^2}$ or $x^2 = \frac{a^2}{y^4}$ $\therefore$ $\frac{a}{y} = \frac{a^2}{y^4}$, $\therefore$ $1 = \frac{a}{y^3}$

or $y = a^{\frac{1}{3}}$ $\therefore$ $x^2 = \dfrac{a}{a^{\frac{1}{3}}} = a^{\frac{2}{3}}$ $\therefore$ $x = a^{\frac{1}{3}}$ and $z = \sqrt{xy} = \sqrt{a^{\frac{1}{3}}a^{\frac{1}{3}}}$ $= \sqrt{a^{\frac{2}{3}}} = a^{\frac{1}{3}}$. Hence it appears that the parallelopipedon is a cube.

The same may easily be solved without impossible roots.

PROB. (14.) TO FIND A POINT P WITHIN A QUADRILATERAL FIGURE $ABCD$, FROM WHICH IF LINES BE DRAWN TO THE ANGULAR POINTS, THE SUM OF THEIR SQUARES SHALL BE THE LEAST POSSIBLE. (Fig. 68.)

Let $AD = b$, $AB = a$, $BC = c$. From the points D, C and P draw straight lines perpendicular to the base or the base produced of the given quadrilateral $\therefore$ $FD = b \sin. A$, $FA = b \cos. A$, $GC = c \sin. B$, $BG = c \cos. B$. Draw EPH parallel to AB and let $AN = x$, $NP = y$ $\therefore$ $EP = FN = AN + AF = x + b \cos. A$, $ED = DF - EF = DF - PN = b \sin. A - y$. $PH = NG = NB + BG = a - x + c \cos. B$, $CH = GC - HG = GC - PN = c \sin. B - y$; we therefore find, $AP^2 = x^2 + y^2$ .... (1.)

$$PB^2 = (a - x)^2 + y^2 \quad \text{.........................} \quad (2.)$$

$$PC^2 = (a - x + c \cos. B)^2 + (c \sin. B - y)^2 \quad (3.)$$

$$DP^2 = (x + b \cos. A)^2 + (b \sin. A - y)^2 \text{......} \quad (4.)$$

Adding these four equations we find;

$AP^2 + PB^2 + PC^2 + DP^2 = 2y^2 + x^2 + (a - x)^2 + (a - x + c \cos. B)^2 + (c \sin. B - y)^2 + (x + b \cos. A)^2 + (b \sin. A - y)^2 = \text{min.}$

First let $y = \text{constant}$ and $x = \text{variable}$, $\therefore$ $x^2 + (a - x)^2 + (a - x + c \cos. B)^2 + (x + b \cos. A)^2 = \text{min.}$ or $4x^2 - 2(2a - b \cos. A + c \cos. B)\, x + 2a^2 + b^2 \cos.^2 A + 2ac \cos. B = 4\left(x^2 - \dfrac{2a - b \cos. A + c \cos. B}{2} x + \dfrac{2a^2 + b^2 \cos.^2 A + 2ac \cos. B}{4}\right) = 4(x^2 - Rx + Q) = \text{min.}$

$\therefore x^2 - Rx + Q = \text{min.} = r, \therefore x = \frac{R}{2} \pm \sqrt{\frac{R^2}{4} + r - Q}.$ Here it is evident that r cannot be taken so small as to make $r - Q$ a negative quantity greater than $\frac{R^2}{4}$, and $\therefore$ when $r = \text{min.}$ we must have $\frac{R^2}{4} = Q - r, \therefore x = \frac{R}{2} = \frac{2a - b\cos. A + c\cos. B}{4}$.

Secondly let $x = \text{constant}$, we find $4y^2 - 2(b\sin. A + c\sin. B)\, y + b^2\sin.^2 A + c^2\sin.^2 B = \text{min.}$ and proceeding exactly in the manner as shown in the case of y being a constant we find $y = \frac{b\sin. A + c\cos. B}{4}$.

The same may be solved without impossible roots.

PROB. (15.) LET $u = ax + by + cz$, A MAXIMUM AND $x^2 + y^2 + z^2 = 1$, FIND x, y, AND $z \therefore u = ax + by + c\sqrt{1 - x^2 - y^2} = \text{MAX.}$

First let $y = \text{constant} \therefore ax + c\sqrt{1 - x^2 - y^2} = a(x + \frac{c}{a}\sqrt{1 - x^2 - y^2}) = \text{max.} \therefore x + \frac{c}{a}\sqrt{1 - x^2 - y^2} = \text{max.} = r \therefore \frac{c^2}{a^2} - \frac{c^2}{a^2}x^2 - \frac{c^2}{a^2}y^2 = r^2 - 2rx + x^2$ and $\therefore \frac{a^2 + c^2}{a^2}x^2 - 2rx = \frac{c^2}{a^2} - \frac{c^2}{a^2}y^2 - r^2$ and therefore $x^2 - \frac{2a^2r}{a^2 + c^2}x = \frac{c^2 - c^2y^2 - a^2r^2}{a^2 + c^2}$

$$\therefore x = \frac{a^2r}{a^2 + c^2} \pm \sqrt{\frac{(c^2 - c^2y^2)(a^2 + c^2) + a^4r^2 - a^2(a^2 + c^2)r^2}{(a^2 + c^2)^2}}$$

and $\therefore$ when $r = \text{max.}$, then $(c^2 - c^2y^2)\,(a^2 + c^2) = a^2c^2r^2$ and $r = \sqrt{\frac{(c^2 - c^2y^2)(a^2 + c^2)}{a^2c^2}}$ and $\therefore$ $x = \frac{a^2\sqrt{\frac{(c^2 - c^2y^2)(a^2 + c^2)}{a^2c^2}}}{a^2 + c^2}$

$$= a\sqrt{\frac{1-y^2}{a^2+c^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.)$$

Secondly, when $x =$ constant, proceeding as above and putting b instead of a and x instead of y we find $y =$

$$\sqrt{\frac{1-x^2}{b^2+c^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.)$$

Squaring equations (1) and (2) we find $x^2 = \dfrac{a^2 - a^2y^2}{a^2+c^2}$

$$y^2 = \frac{a^2-(a^2+c^2)x^2}{a^2} \text{ and } \therefore y^2 = \frac{b^2-b^2x^2}{b^2+c^2} = \frac{a^2-(a^2+c^2)x^2}{a^2}$$

$$= 1 - \frac{(a^2+c^2)x^2}{a^2} \therefore \frac{(a^2+c^2)x^2}{a^2} = 1 - \frac{b^2-b^2x^2}{b^2+c^2} = \frac{c^2+b^2x^2}{b^2+c^2}$$

$$\frac{c^2}{b^2+c^2} + \frac{b^2}{b^2+c^2}x^2 \therefore \left(\frac{a^2+c^2}{a^2} - \frac{b^2}{b^2+c^2}\right)x^2 = \frac{c^2}{b^2+c^2}$$

$$\therefore \frac{a^2b^2 + a^2c^2 + b^2c^2 + c^4 - a^2b^2}{a^2(b^2+c^2)}x^2 = \frac{c^2}{b^2+c^2} \therefore x^2 =$$

$$\frac{a^2}{a^2+b^2+c^2} \therefore x = \frac{a}{\sqrt{a^2+b^2+c^2}} \text{ and } \therefore y^2 = \frac{b^2-b^2x^2}{b^2+c^2} =$$

$$\frac{b^2 - \dfrac{a^2b^2}{a^2+b^2+c^2}}{b^2+c^2} = \frac{b^4+b^2c^2}{(b^2+c^2)\ (a^2+b^2+c^2)} = \frac{b^2}{a^2+b^2+c^2}$$

$$\therefore x = \frac{b}{\sqrt{a^2+b^2+c^2}} \text{ and } z^2 = 1 - x^2 - y^2 = 1 - \frac{a^2}{a^2+b^2+c^2}$$

$$\frac{b^2}{a^2+b^2+c^2} = \frac{c^2}{a^2+b^2+c^2} \therefore z = \frac{c}{\sqrt{a^2+b^2+c^2}}.$$

The same may easily be solved without impossible roots.

PROB. (16.) FIND THAT POINT WITHIN A GIVEN TRIANGLE, FROM WHICH IF LINES BE DRAWN TO THE ANGULAR POINTS, THE SUM OF THEIR SQUARES SHALL BE A MINIMUM. (Fig. 69.)

This Problem is a more elegant solution of Prob. (8.)

Let ABC be the triangle, and P a point within it; a, b, c the sides of the triangle. Draw PN, AD perpendicular to the base; join AP, BP, CP. Let $CN = x$; $NP = y$; then $AD = b \sin. C$; $CD = b \cos. C$. Then $CP^2 = x^2 + y^2$, $BP^2 = y^2 + (a - x)^2 = y^2 + x^2 + a^2 - 2ax$, $AP^2 = (b \cos. C - x)^2 + (b \sin. C - y)^2 = b^2 + x^2 + y^2 - 2b(x \cos. C + y \sin. C)$; $3x^2 + 3y^2 + a^2 + b^2 - 2ax - 2b\,(x \cos. C + y \sin. C) =$ min. $\therefore\ x^2 + y^2 + \dfrac{a^2 + b^2}{3} - \dfrac{2ax}{3} - \dfrac{2b}{3}(x \cos. C + y \sin. C) =$ min.

First let $y =$ constant $\therefore\ x^2 + \dfrac{a^2 + b^2}{3} - \dfrac{2a + 2b \cos. C}{3} x =$ min. $= r$ $\therefore\ x^2 - \dfrac{2a + 2b \cos. C}{3} x = r - \dfrac{a^2 + b^2}{3}$ and $x = \dfrac{a + b \cos. C}{3} \pm \sqrt{r - \dfrac{a^2 + b^2}{3} + \dfrac{(a + b \cos. C)^2}{9}}$ $= \dfrac{a + b \cos. C}{3} \pm \sqrt{r + \dfrac{2ab \cos. C + b^2 \cos.^2 C - 2a^2 - 3b^2}{9}}$.

It is evident that if $a \,7\, b$, then $a \,7\, b \cos. C$ $\therefore\ 2a^2 \,7\, 2ab \cos. C$ $\therefore\ 2ab \cos. C - 2a^2$ is negative and $b^2 \cos.^2 C$ is evidently $\angle\ 3b^2$ $\therefore\ \dfrac{2ab \cos. C + b^2 \cos.^2 C - 2a^2 - 3b^2}{9}$ is negative $= -P$ $\therefore\ x = \frac{1}{3}(a + b \cos. C) \pm \sqrt{r - P}$ $\therefore$ when $r =$ min. then $r = P$ $\therefore\ x = \frac{1}{3}(a + b \cos. C)$.

Secondly when $x =$ constant, then we shall have $y^2 + \dfrac{a^2 + b^2}{3} - \dfrac{2b \sin. C}{3} y =$ min. $= r$ $\therefore$

$$y = \frac{b\sin.C}{3} \pm \sqrt{r - \frac{a^2+b^2}{3} + \frac{b^2\sin.^2C}{9}}$$

$$= \frac{b\sin.C}{3} \pm \sqrt{r + \frac{b^2\sin.^2C - 3b^2}{9} - \frac{a^2}{3}}.$$ Here it is evident that $\frac{b^2\sin.^2C - 3b^2}{9} = \frac{b^2}{9}(\sin.^2C - 3) =$ a negative quantity which let $= -Q \therefore y = \frac{b\sin.C}{3} \pm \sqrt{r - Q},$ $\therefore$ when $r =$ min. we must have $r = Q \therefore y = \frac{b\sin.C}{3}$.

The same may easily be solved without impossible roots.

For $-Q$ read $-Q-\frac{a^2}{3}$—Ed.

PROB. (17.) TO FIND A POINT WITHIN A GIVEN TRIANGLE, FROM WHICH IF PERPENDICULARS BE LET FALL UPON THE SIDES, THE SUM OF THEIR SQUARES SHALL BE A MINIMUM. (Fig. 70.)

Let ABC be the triangle as before, P the point within it, draw PN, PM, PQ respectively perpendicular to CB, CA AB. Let $CN = x$; $NP = y$, $PM = p$, $PQ = q$, $CB = a$, $CA = b$, $AB = c \therefore u = y^2 + p^2 + q^2$. Now it is evident that $p^2 = MP^2 = FP^2 \times \cos.^2 MPF = FP^2 \cos.^2 C = (FN - PN)^2 \cos.^2 C = (x \tan. C - y)^2 \cos.^2 C = \frac{(x\tan. C - y)^2}{\sec.^2 C} = \left(\frac{y - x\tan. C}{\sec. C}\right)^2 = (y\cos. C - x\sin. C)^2$ Also $q^2 = PQ^2 = PE^2 \cos.^2 EPQ = (EN - PN)^2 \cos.^2 B = \left(\frac{y - (a-x)\tan. B}{\sec. B}\right)^2 = \{y\cos. B - (a-x)\sin. B\}^2 \therefore$ $u = y^2 + (y\cos. C - x\sin. C)^2 + \{y\cos. B - (a-x)\sin. B\}^2$ $=$ min. or $y^2 + y^2\cos.^2 C - 2xy\cos. C\sin. C + y^2\cos.^2 B + x^2\sin.^2 C - 2y(a-x)\cos. B\sin. B + (a-x)^2\sin.^2 B$ $=$ min.

First let x = constant $\therefore y^2 + y^2 \cos.^2 C - 2xy \cos. C \sin. C + y^2 \cos.^2 B - 2y\,(a - x) \cos. B \sin. B$

$$= (1 + \cos.^2 C + \cos.^2 B) y^2 - 2y\{x \cos. C \sin. C + (a - x) \cos. B \sin. B\}$$

$$= (1 + \cos.^2 C + \cos.^2 B)\left(y^2 - 2y\frac{\{x \cos. C \sin. C + (a - x) \cos. B \sin. B\}}{1 + \cos.^2 C + \cos.^2 B}\right)$$

$= \text{min.} \therefore y^2 - 2\dfrac{\{x \cos. C \sin. C + (a - x) \cos. B \sin. B\}}{1 + \cos.^2 C + \cos.^2 B} y =$ min. = a negative quantity and $\therefore$ as in the foregoing problem we find $y = \dfrac{x \cos. C \sin. C + (a - x) \cos. B \sin. B}{1 + \cos.^2 C + \cos.^2 B}$

$$= \frac{(\cos. C \sin. C - \cos. B \sin. B)\, x + a \cos. B \sin. B}{1 + \cos.^2 C + \cos.^2 B} \quad (1)$$

Secondly let y = constant $\therefore -2xy \cos. C \sin. C + x^2 \sin.^2 C$

$$+ 2yx \cos. B \sin. B - 2ax \sin.^2 B + x^2 \sin.^2 B$$

$(\sin.^2 B + \sin.^2 C) x^2 - 2(y \cos. C \sin. C - y \cos. B \sin. B + a \sin.^2 B) x =$

$$(\sin.^2 B + \sin.^2 C)\left(x^2 - 2\,\frac{(y \cos. C \sin. C - y \cos. B \sin. B + a \sin.^2 B)}{\sin.^2 B + \sin.^2 C} x\right)$$

$= \text{min.} \therefore x^2 - \dfrac{2(y \cos. C \sin. C - y \cos. B \sin. B + a \sin.^2 B)}{\sin.^2 B + \sin.^2 C} x =$ min. = a negative quantity, and $\therefore$ as in the foregoing problem, we find $x = \dfrac{y(\cos. C \sin. C - \cos. B \sin. B) + a \sin.^2 B}{\sin.^2 B + \sin.^2 C} \ldots\ldots (2)$

$$\left.\begin{array}{l} \text{Now let } \cos. C \sin. C - \cos. B \sin. B = P \\ a \cos. B \sin. B = S \\ 1 + \cos.^2 C + \cos.^2 B = T \\ a \sin.^2 B = Q \\ \text{Sin.}^2 B + \sin.^2 C = R \end{array}\right\} \ldots\ldots\ldots \quad (3.)$$

$$\therefore x = \frac{Py + Q}{R} \ldots (4.) \text{ and } y = \frac{Px + S}{T} \text{ or } x = \frac{Ty - S}{P} \ldots (5.)$$

Comparing equations (4) and (5) we find,

$y = \dfrac{RS + PQ}{RT - P^2}$ and substituting the values of R, S, P, Q, T from equations (3) we find,

$$y=\frac{(\sin.^2B+\sin.^2C)\,a\cos.B\sin.B+(\cos.C\sin.C-\cos.B\sin.B)\,a\sin.^2B}{(1+\cos.^2B+\cos.^2C)(\sin.^2C+\sin.^2B)-(\cos.C\sin.C-\cos.B\sin.B)^2}$$

$$=\frac{a\sin.^2C\sin.B\cos.B+a\sin.^2B\cos.C\sin.C}{\sin.^2B+\sin.^2C+\sin.^2C\cos.^2B+\cos.^2C\sin.^2B+2\cos.C\sin.C\cos.B\sin.B}$$

$$\frac{a\sin.B\sin.C\sin.(B+C)}{1-\cos.^2B+1-\cos.^2C+\sin.^2C\cos.^2B+\cos.^2C\sin.^2B+2\cos.C\sin.C\cos.B\sin.B}$$

$$=\frac{a\sin.A\sin.B\sin.C}{2(1-\cos.^2B\cos.^2C+\cos.B\cos.C\sin.B\sin.C)},\text{ and}$$

substituting the values of sines and cosines of $A,B,C,$ in terms of the sides of the given triangle we find, $y=\frac{abc\sin.A}{a^2+b^2+c^2}\therefore p=\frac{abc\sin.B}{a^2+b^2+c^2}$ and $q=\frac{abc\sin.C}{a^2+b^2+c^2}$.

The same may easily be solved without impossible roots.

PROB. (18.) TO FIND THE VALUES OF $x,y,z,$ THAT, $ax^2y^3z^4-x^3y^3z^4-x^2y^4z^4-x^2y^3z^5$ MAY BE = MAX.

First let x, y = constants and z = variable,

$\therefore ax^2y^3z^4-x^3y^3z^4-x^2y^4z^4-x^2y^3z^5$

$=x^2y^3\{(a-x-y)z^4-z^5\}$ = max. and

$\therefore (a-x-y)z^4-z^5$ = max. $\therefore$ by Prob. (10), chap. 3,

$$z=\frac{4(a-x-y)}{5}\text{ or }4x+4y+5z=4a\ \ldots\ldots\ldots\ldots\ (1)$$

Secondly let x, z = constants and y = variable, then proceeding as before we find $(a-x-z)y^3-y^4$ = max. and

$$\therefore y=\frac{3(a-x-z)}{4}\therefore 3x+4y+3z=3a\ \ldots\ldots\ldots\ (2.)$$

Thirdly let y, z = constants, and x = variable, then as before $(a-y-z)x^2-x^3$ = max. $\therefore$ by Prob. (2), chap. 2,

$x = \frac{2(a - y - z)}{3}$ and $\therefore 3x + 2y + 2z = 2a$ .... (3.)

Subtracting (3) from (2) we find $2y + z = a$ .... (4.)

Multiplying equations (1) and (3) by 3 and 4 respectively we find, $12x + 12y + 15z = 12a$

$$\begin{array}{r} 12x + 8y + 8z = 8a \\ \hline 4y + 7z = 4a, \end{array}$$ and multiplying

(4) by 2 we find $4y + 2z = 2a$

$$5z = 2a \therefore z = \frac{2a}{5} \therefore 4y + \frac{4a}{5} = 2a$$

$\therefore y = \frac{3a}{10}$ and $\therefore 3x + 2y + 2z = 3x + \frac{3a}{5} + \frac{4a}{5} = 3x + \frac{7a}{5} = 2a \therefore 3x = 2a - \frac{7a}{5} = \frac{3a}{5} \therefore x = \frac{a}{5}$.

The same may be solved without impossible roots.

SUPPLEMENT.

IT will be observed throughout this work that a great many equations of the second degree solved for finding out the maximum value of r have been reduced to the form $x^2 + Ax = -r$ or $x^2 + Ax + r = 0$, where A is generally negative, and in like manner the cubic and biquadratic equations have been reduced to the forms, $x^3 + Ax^2 + Bx + r = 0$, $x^4 + Ax^3 + Bx^2 + Cx + r = 0$, where the maximum value of r is to be determined.

The object of this supplement is to solve these general equations, and thus to find out general expressions which may enable us to solve numerous problems of this book in an instant, without going through long and sometimes tedious operations.

We will also add in this part of the work a few interesting problems which we have unfortunately forgotten to put in their proper places.

1st. Solve the equation $x^2 + Ax + r = 0$, where $r =$ max. We have $x^2 + Ax = -r \therefore x = -\frac{A}{2} + \sqrt{\frac{A^2}{4} - r}$

$\therefore$ when $r =$ max. we must have $\frac{A^2}{4} = r \therefore x = -\frac{A}{2}\ldots$ (A.)

EX. $20x - x^2 =$ max. $= r \therefore x^2 - 20x + r = 0$. Here $A = -20 \therefore$ by (A), $x = -\frac{-20}{2} = 10$.

In like manner other examples of this kind may be solved by means of (A).

2nd. Solve the cubic equation, $x^3 + Ax^2 + Bx + r = 0$. Let a negative root of this equation $= a \therefore$

$$x+a \,)\, x^3+Ax^2+Bx+r=0 \,(\, x^2+(A-a)x+a^2+B-aA=0\ (1.)$$

$$\begin{array}{l} x^3+ax^2 \\ \hline (A-a)\,x^2+Bx \\ (A-a)\,x^2+(aA-a^2)x \\ \hline (a^2+B-aA)\,x+r \\ (a^2+B-aA)\,x+a\,(a^2+B-aA) \\ \hline \end{array}$$

$\therefore\ a^2+B-aA=\frac{r}{a}$ $\therefore$ Equa. (1) gives $x^2+(A-a)\,x$ $=-\frac{r}{a}$, $\therefore\ x=-\frac{A-a}{2}+\sqrt{\frac{(A-a)^2}{4}-\frac{r}{a}}$.

Here it is evident that when $r=$ max. we must have $\frac{(A-a)^2}{4}=\frac{r}{a}=a^2+B-aA$, $\therefore\ A^2-2aA+a^2=4a^2$ $+4B-4aA$ or $3a^2-2Aa=A^2-4B$ or $a-\frac{2A}{3}\,a=$ $\frac{A^2-4B}{3}$, $\therefore\ a=\frac{A+\sqrt{4A^2-12B}}{3}$ and $x=-\frac{A-a}{2}=$ $\frac{a-A}{2}=\frac{\sqrt{4A^2-12B}-2A}{6}$ .... (B.)

Ex. (1) $x^3-x^2+r=0$. Here $A=-1$, $B=0$, $\therefore\ x$ $=\frac{1+1}{3}=\frac{2}{3}$. Ex. (2) $x^3-x+r=0$, $A=0$, $B=-$ 1, $\therefore\ x=\frac{\sqrt{3}}{3}=\frac{1}{\sqrt{3}}$. Ex. (3) $x^3-6x-15x+r=0$, $A=-6$, $B=-15$, $\therefore$ by (B), $x=\frac{\sqrt{36+45}+6}{3}=5$.

3rd. Solve the general equation of the fourth degree, viz. $x^4+Ax^3+Bx^2+Cx+r=0$.

Let the product of the two values of this equation $=x^2$ $+ax+b$, and we therefore find,

$$x^2+ax+b \,)\, x^4+Ax^3+Bx^2+Cx+r=0 \,(\, x^2+(A-a)x+B+a^2-Aa-b=0\ldots(1.)$$

$$\begin{array}{l}
x^4+ax^3+bx^2 \\
\hline
(A-a)\,x^3+(B-b)\,x^2+Cx \\
(A-a)\,x^3+(Aa-a^2)\,x^2+(Ab-ab)x \\
\hline
(B+a^2-Aa-b)\,x^2+(C+ab-Ab)\,x+r \\
(B+a^2-Aa-b)x^2+(aB+a^3-Aa^2-ab)x+b\,(B+a^2-Aa-b) \\
\hline
\end{array}$$

$\therefore B + a^2 - Aa - b = \frac{r}{b}$ .... (2.)

Also $C + ab - Ab = aB + a^3 - Aa^2 - ab$, $\therefore b = \frac{aB + a^3 - Aa^2 - C}{2a - A}$ .... (3.)

Now solving the equation (1) we find $x = -\frac{A-a}{2} + \sqrt{\frac{(A-a)^2}{4} - \frac{r}{b}}$; and here it is evident that when $r =$ max. then $\frac{(A-a)^2}{4} = \frac{r}{b} = B + a^2 - Aa - b$, $\therefore (A-a)^2 = 4B + 4a^2 - 4Aa - 4b$, and from (3)

$$(A-a)^2 = 4B + 4a^2 - 4Aa - \frac{4aB + 4a^3 - 4Aa^2 - 4C}{2a - A}$$

$$= \frac{4aB + 4a^3 - 8Aa^2 - 4AB + 4A^2a + 4C}{2a - A}$$

or $4aB + 4a^3 - 8Aa^2 - 4AB + 4A^2a + 4C = 4A^2a - 5a^2A + 2a^3 - A^3$, and therefore $a^3 - \frac{3A}{2}a^2 + 2Ba - 2AB + 2C + \frac{A^3}{2} = 0$ .... (C.)

Now it is evident that from this equation the value of a may be determined, which, when put in $x = -\frac{A-a}{2} = \frac{a-A}{2}$, we will find out the value of x sought.

Ex. (1). $x^4 - x^3 + r = 0$. $A = -1$, $B = 0$, $C = 0$, $\therefore$ $a^3 + \frac{3}{2}a^2 - \frac{1}{2} = 0$. Let $a = \frac{1}{2}$, $\therefore$ $a^3 + \frac{3}{2}a^2 - \frac{1}{2} = \frac{1}{8} + \frac{3}{8} - \frac{1}{2} = 0$, $\therefore$ $x = \frac{a - A}{2} = \frac{\frac{1}{2} + 1}{2} = \frac{3}{4}$.

Ex. (2). $x^4 - x + r = 0$, $A = 0$, $B = 0$, and $C = -1$, $\therefore$ $a^3 - 2 = 0$, $\therefore$ $a = 2$ and $x = -\frac{A - a}{2} = \frac{a}{2} = \frac{2^{\frac{1}{3}}}{2} = \frac{1}{2^{\frac{2}{3}}} - \frac{1}{\sqrt[3]{4}}$.

Ex. (3). $x^4 - 8x^3 + 22x^2 - 24x + r = 0$. Here $A = -8$, $B = 22$, $C = -24$, $\therefore$ $a^3 + 12a^2 + 44a + 48 = 0$. Let $a = -4$, $\therefore$ $-64 + 192 + 48 - 176 = -48 + 48 = 0$, and $x = \frac{a - A}{2} = \frac{a + 8}{2} = \frac{-4 + 8}{2} = 2$.

Ex. (4). To inscribe the greatest parabola in a given isosceles triangle. (Fig. 62.)

Let $AD = b$, $GD = a$, $GP = x$, $\therefore$ the area of the parabola $= \frac{4b}{3a}\sqrt{(a - x)^3 x} = \text{max.}$ $\therefore$ $(a - x)^3 x = a^3 x - 3a^2 x^2 + 3ax^3 - x^4 = \text{max.} = r$, $\therefore$ $x^4 - 3ax^3 + 3a^2x^2 - a^3 x + r = 0$. Here $A = -3a$, $B = 3a^2$, $C = -a^3$. Now substituting these values of A, B, C, and putting y instead of a in the equation (C) we find,

$y^3 + \frac{9a}{2}y^2 + 6a^2 y + \frac{5a^3}{2} = 0$. By trial the value of y is found $= -\frac{5a}{2}$, $\therefore$ $x = \frac{-A + y}{2}$

$$= \frac{3a - \frac{5a}{2}}{2} = \frac{a}{4}.$$

Ex. (5). In the trapezium $ABCD$, the base $AB = a$, $AD = BC = b$, find CD, CD being parallel to AB, that the area may be a maximum (m & n are the points where the perps. cut the parallel line required and $mn = x$).

It is evident that $Am = nB$ $\therefore$ the area of the whole trapezium $= \frac{Dm \times Am}{2} + mn \times Dm + \frac{Cn \times nB}{2} = \frac{Dm \times Am}{2}$ $+ mn \times Dm + \frac{Dm \times Am}{2} = Dm \times Am + mn \times Dm =$

$$\frac{a-x}{2}\sqrt{b^2 - \left(\frac{a-x}{2}\right)^2} + x\sqrt{b^2 - \left(\frac{a-x}{2}\right)^2}$$

$$= \frac{a+x}{2}\sqrt{b^2 - \left(\frac{a-x}{2}\right)^2} = \sqrt{b^2\left(\frac{a+x}{2}\right)^2 - \left(\frac{a^2-x^2}{4}\right)^2}$$

$=$ max. $= r$, $\therefore$ $x^4 - 2(2b^2 + a^2)x^2 - 8ab^2x - a^2(4b^2 - a^2)$ $+ r = 0$. Here $A = 0$, $B = -2(2b^2 + a^2)$, $C = -8ab^2$, $\therefore$ $y^3 - 4(2b^2 + a^2)y - 16ab^2 = 0$. Let $y = -2a$, $\therefore$ $-8a^3 + 16ab^2 + 8a^3 - 16ab^2 = 0$, and therefore $\frac{y^3 - 4(2b^2 + a^2)y - 16ab^2}{y + 2a} = y^2 - 2ay - 8b^2 = 0$, $\therefore$ y $= a + \sqrt{8b^2 + a^2}$ and $x = \frac{y - A}{2} = \frac{y}{2} = \frac{a + \sqrt{8b^2 + a^2}}{2}$.

It may be remarked in this place that cubic equations got by reduction of biquadratic equations may be solved by Cardon's Rule, instead of the method of trial as effected in the preceding examples.

A FEW NEW PROBLEMS.

PROB. (1.) FIND THE GREATEST AREA THAT CAN BE INCLUDED BY FOUR GIVEN STRAIGHT LINES. (Fig. 71.)

Let $a, b, c, d,$ = four given straight lines, n = the angle included by a, b and m = the angle included by c, d and D = diagonal; $\therefore$ area required $= \dfrac{cd\sin.m}{2} + \dfrac{ab\sin.n}{2} = \dfrac{cd}{2}\left(\sin.m + \dfrac{ab}{cd}\sin.n\right) = \max.$ $\therefore$ $\sin.m + \dfrac{ab}{cd}\sin.n = \max.$

Squaring this expression, $\sin.^2 m + \dfrac{2ab}{cd}\sin.m\sin.n + \dfrac{a^2b^2}{c^2d^2}\sin.^2 n = \max. = r$ .... (1.)

But $c^2 + d^2 - 2cd\cos.m = D^2 = a^2 + b^2 - 2ab\cos.n,$

$\therefore$ $\cos.m - \dfrac{ab}{cd}\cos.n = \dfrac{c^2 + d^2 - a^2 - b^2}{2cd} = B$ and $\therefore$

$\cos.^2 m - \dfrac{2ab}{cd}\cos.m\cos.n + \dfrac{a^2b^2}{c^2d^2}\cos.^2 n = B^2$ .... (2.)

Adding equations (1) and (2), and transposing, we find

$$-\cos.(m+n) = \frac{r + B^2 - \dfrac{a^2b^2}{c^2d^2} - 1}{\dfrac{2ab}{cd}};$$ and here it is evident

that the greatest value for r, or the second member of the equation, is the greatest positive value of the first member; that is to say, we must have $-\cos.(m+n) = -1 \times \cos.(m+n) = 1$, which can only take place when $\cos.(m+n) = -1$ or $m + n = 180^\circ$ $\therefore$ $\sin.m = \sin.n$, and therefore

Area $= \dfrac{ab + cd}{2}\sin.n = \sqrt{(P - a)(P - b)(P - c)(P - d)}$

where $P = \dfrac{a + b + c + d}{2}$ as found by calculation.

PROB. (2.) TO FIND SUCH A VALUE OF x THAT $(mx + n)(ny + m) =$ MAX. AND $a^{mx}.\ y^{ny} = c$.

From the second equation we find, $mx \log a + ny \log b = \log c$. Let $\log a = A$, $\log b = B$, and $\log c = C$, $\therefore mAx + nBy = C$, $\therefore ny = \dfrac{C - mAx}{B}$ $\therefore ny + m = \dfrac{C - mAx + mB}{B}$, and therefore $(mx + n)(ny + m) = \dfrac{m^2Ax^2 - (mC + m^2B - nmA)}{B}x - \dfrac{nC + nmB}{B} = -r$, and therefore $x^2 - \dfrac{(mC + m^2B - nmA)}{m^2A}x = \dfrac{nC + nmB}{m^2A} - \dfrac{Br}{m^2A}$, $\therefore$ we find (as in problems in Chap. 1st) $x = \dfrac{C + mB - nA}{2mA} = \dfrac{\log c + m \log b - n \log a}{2m \log a} = \dfrac{\log \dfrac{cb^m}{a^n}}{\log a^{2m}}$.

PROB. (3.) *OM* AND *OP* ARE TWO ARCS OF GREAT CIRCLES ON A SPHERE, AND THE ARC *PM* IS DRAWN PERPENDICULAR TO *OM*, FIND WHEN THE DIFFERENCE BETWEEN *OP* AND *OM* IS THE GREATEST. (Fig. 72.)

Let $POM = \alpha$, $OP = \phi$, and $OM = \theta$, $\therefore \phi - \theta =$ max. $= r$. By Napier's Rules for the solution of right-angled triangles (spherical) $\tan\theta = \cos.\alpha \tan\phi$, $\therefore \theta = \phi - r$ $\therefore$

$\tan\theta = \dfrac{\tan\phi + \tan r}{1 - \tan\phi \tan r} = \cos.\alpha \tan\phi$ or $\dfrac{x + r'}{1 - r'x}$ (where $r' = \tan r = \text{max.}) = ax \times$ (where $a = \cos.\alpha$, and $x = \tan\phi$) or $x^2 + \dfrac{1-a}{ar'}x = -\dfrac{1}{a}$, $\therefore x = \dfrac{a-1}{2ar'} \pm \sqrt{\dfrac{(1-a)^2}{4a^2r^2} - \dfrac{1}{a}}$. Here it is evident that when $r = \text{max.}$ then $\dfrac{(1-a)^2}{4a^2r^2} = \text{min.}$ $\therefore$ when $r = \text{max.}$ then we must have $\dfrac{(1-a)^2}{4a^2r^2} = \dfrac{1}{a}$, $\therefore r = \dfrac{a-1}{2\sqrt{a}}$ $\therefore x = \dfrac{a-1}{2ar} = \dfrac{1}{\sqrt{a}} = (\cos.\alpha)^{-\frac{1}{2}}$.

I had to say something more regarding the Algebraical theory of Maxima and Minima, but being afraid of enlarging the work too much, I conclude these sheets.

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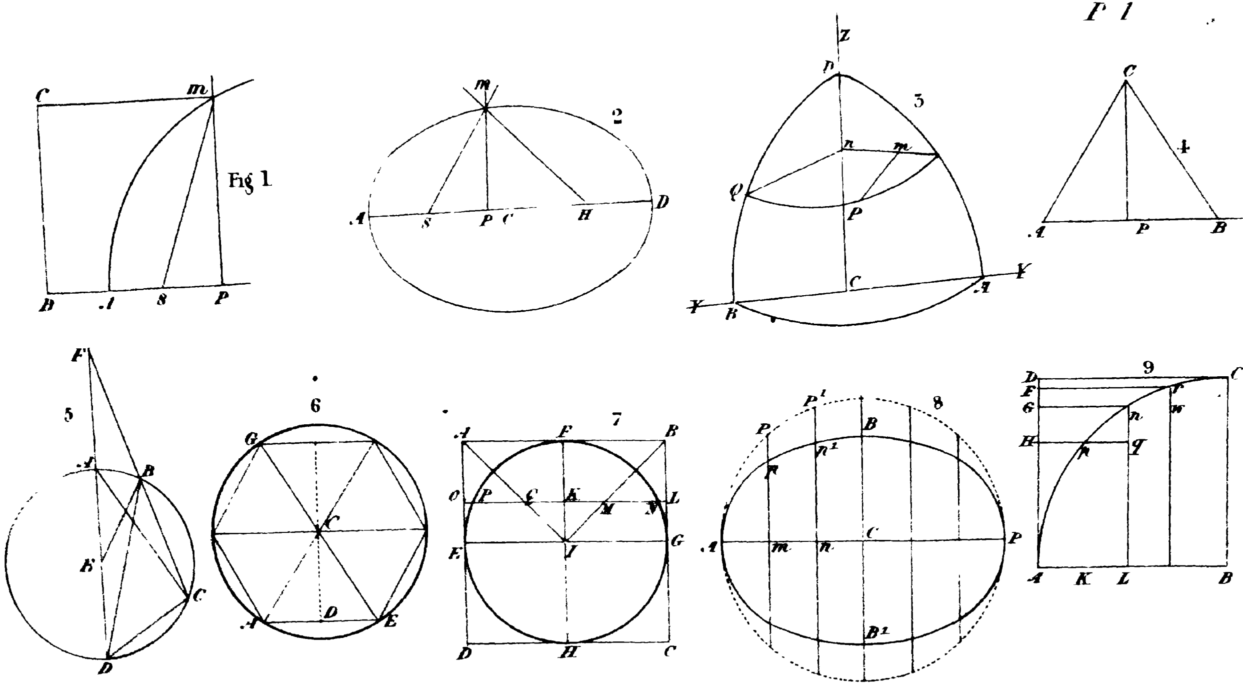

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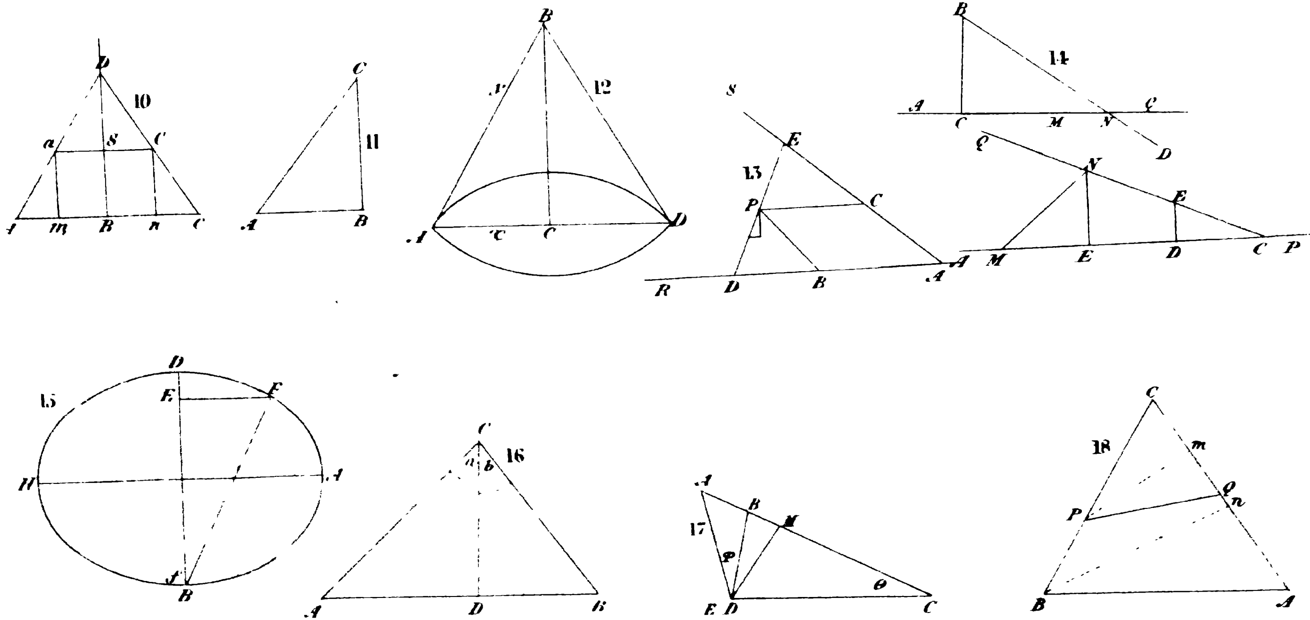

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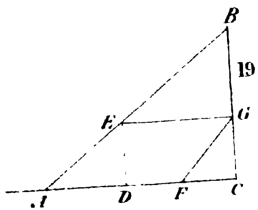

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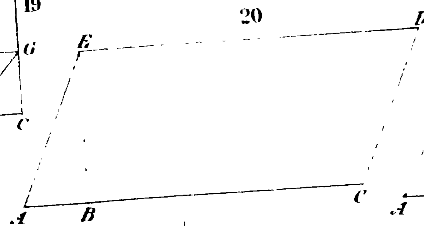

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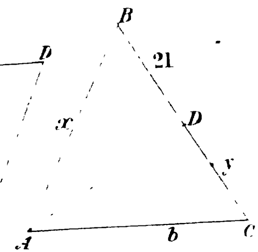

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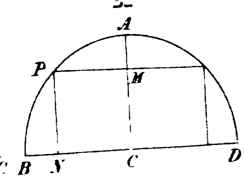

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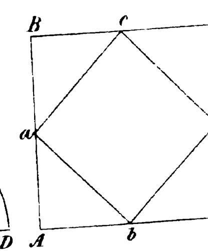

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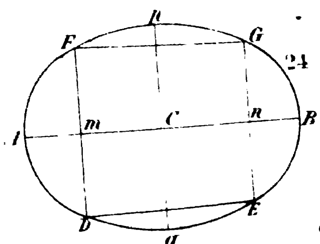

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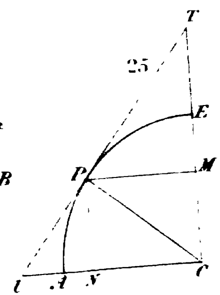

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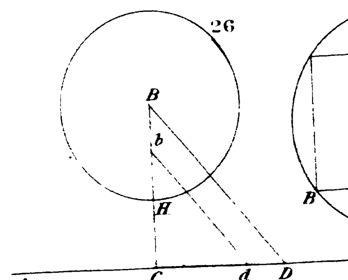

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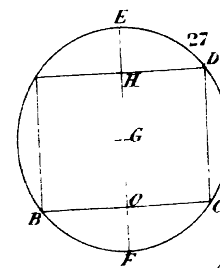

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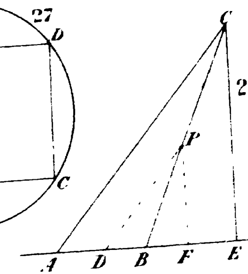

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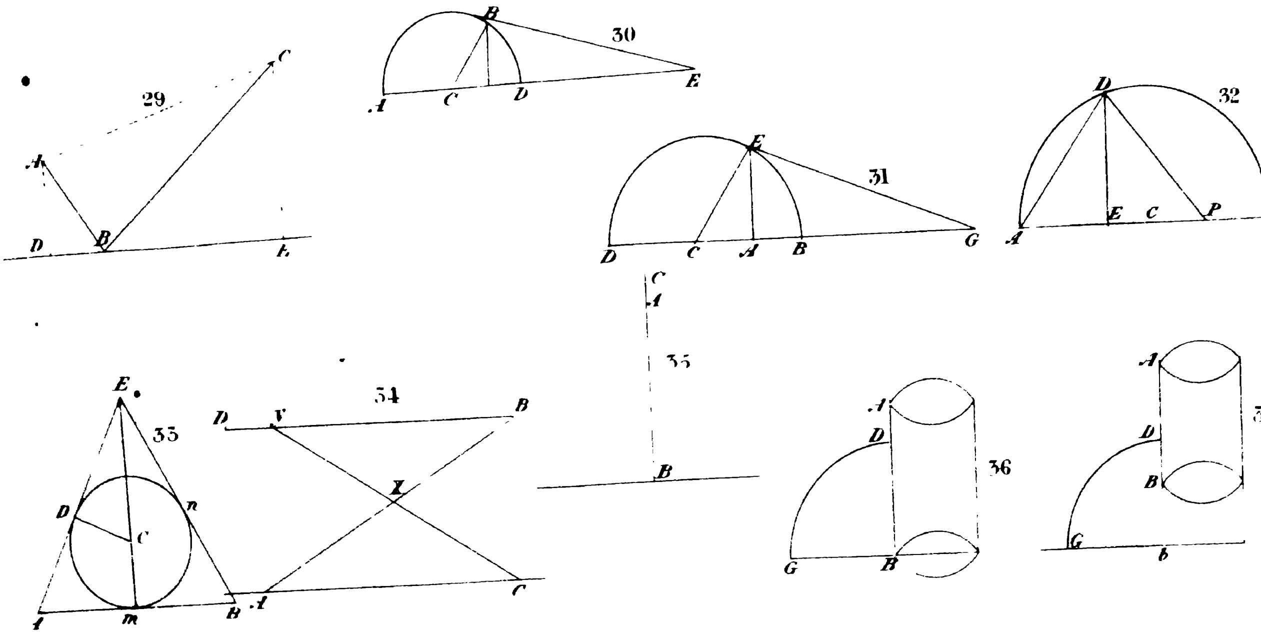

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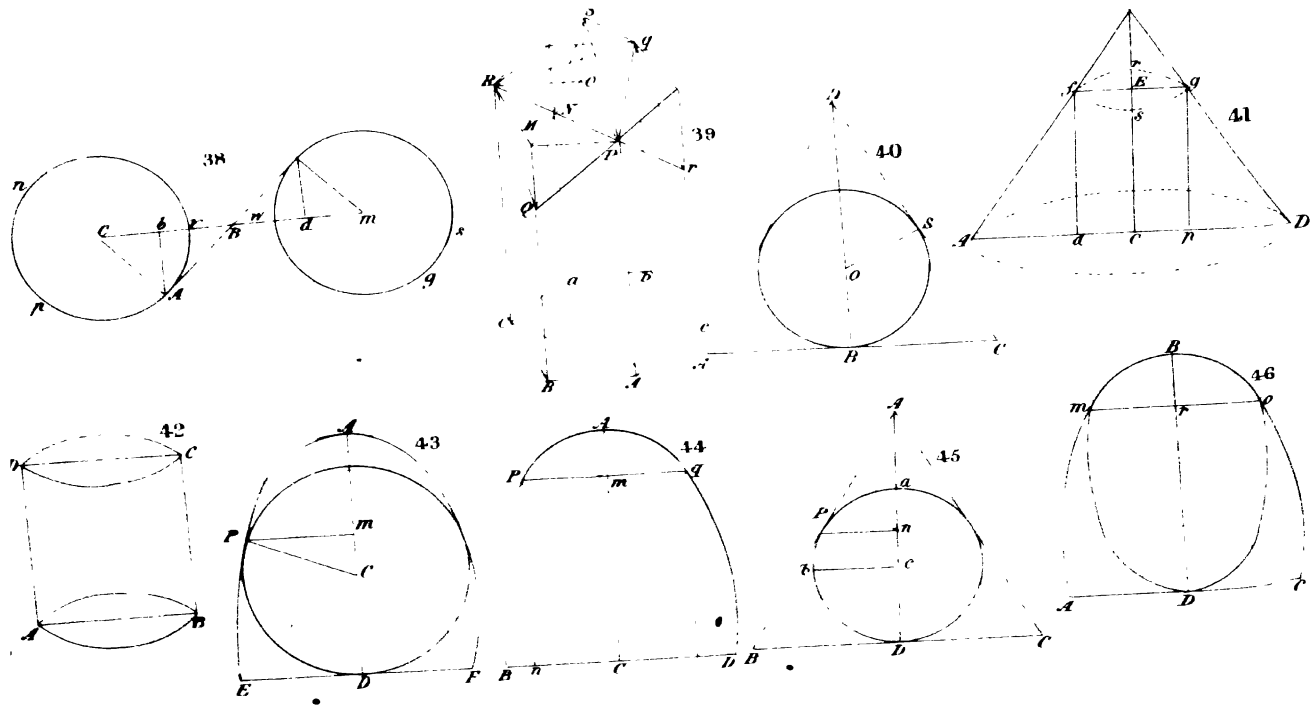

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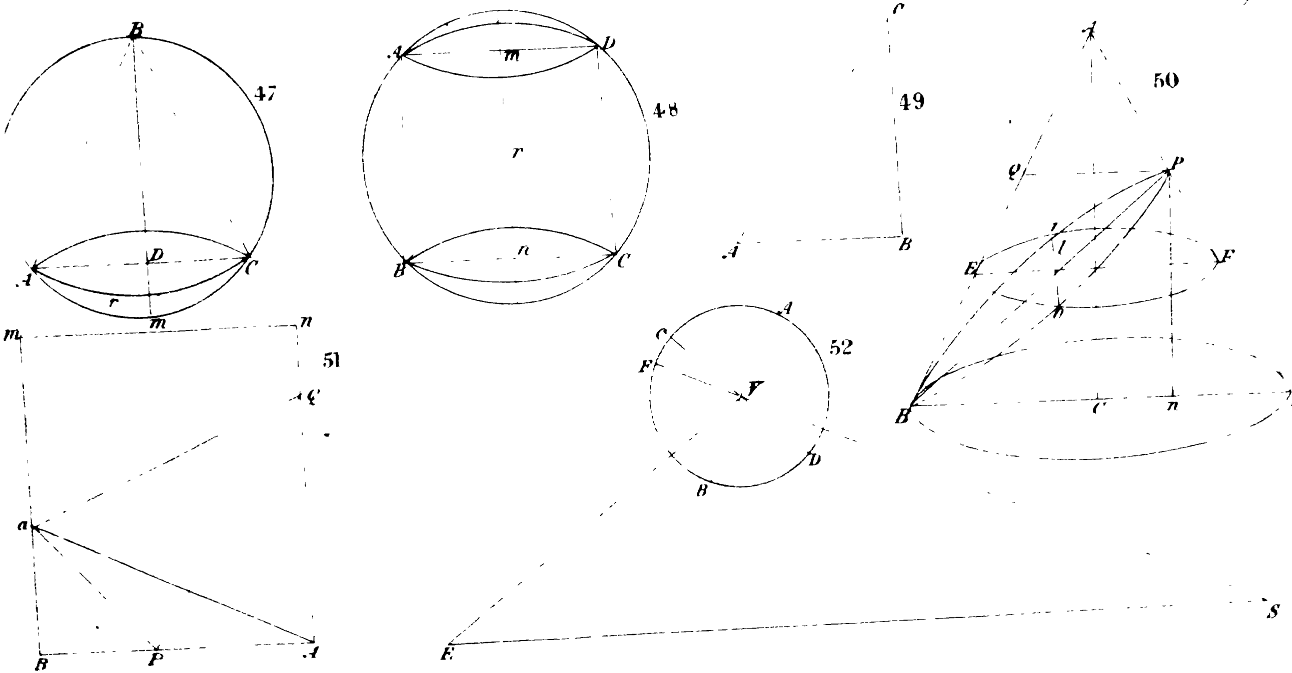

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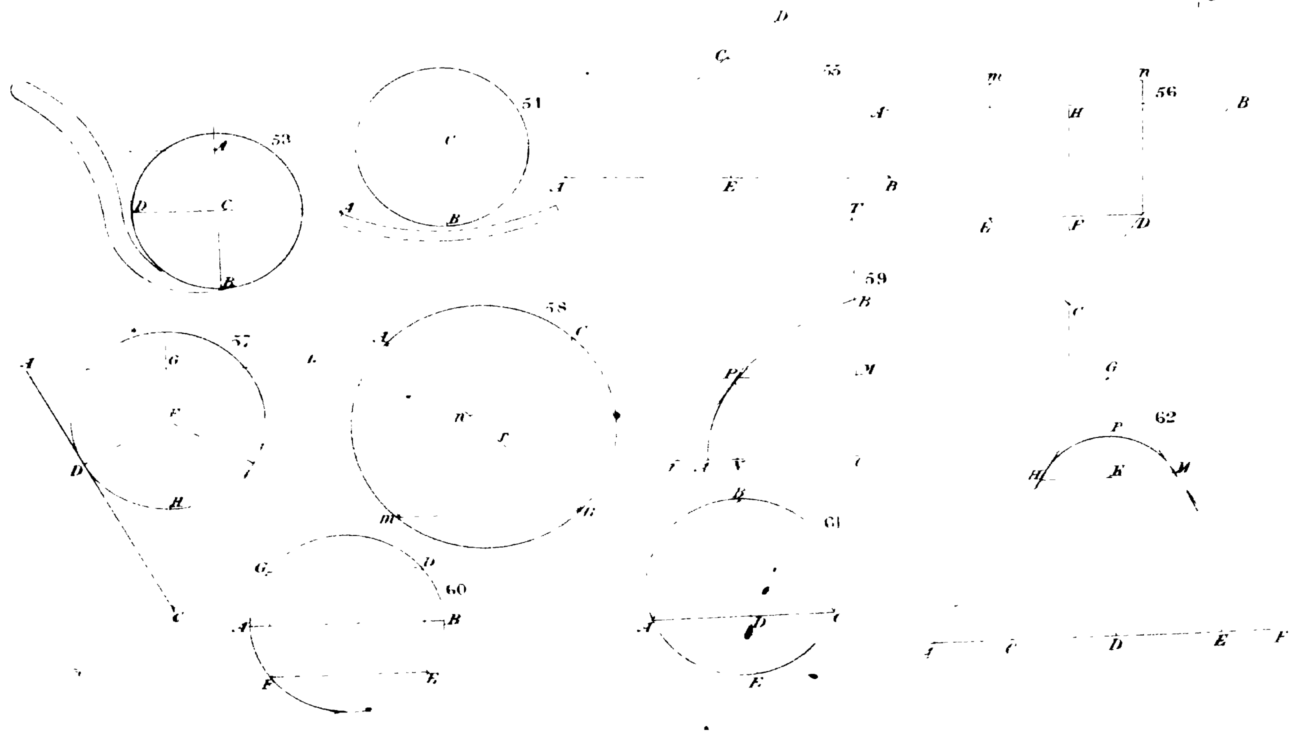

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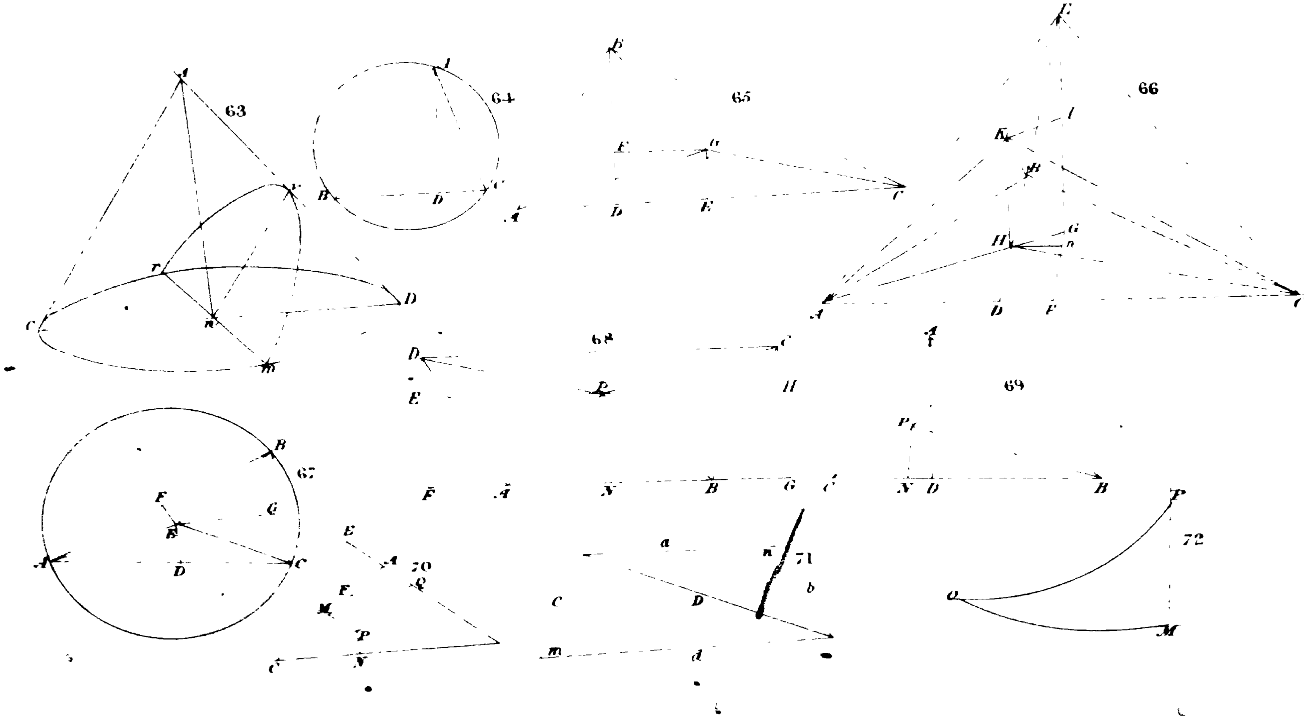

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